

GLOBAL F-SPLITTING RATIO OF MODULES

ALESSANDRO DE STEFANI, THOMAS POLSTRA, AND YONGWEI YAO

ABSTRACT. Techniques are developed to extend the notions of F-splitting ratios to modules over rings of prime characteristic, which are not assumed to be local. We first develop the local theory for F-splitting ratio of modules over local rings, and then extend it to the global setting. We also prove that strong F-regularity of a pair (R, \mathcal{D}) , where \mathcal{D} is a Cartier algebra, is equivalent to the positivity of the global F-signature $s(R, \mathcal{D})$ of the pair. This extends a result previously proved by these authors, by removing an extra assumption on the Cartier algebra.

1. INTRODUCTION

This article is focused on extending the notion of *F-splitting ratio* of a local ring in two directions: from the local to the global setting, and from the ring to all finitely generated R -modules. The F-splitting ratio of a local ring, denoted $r_F(R)$, is a measure of the asymptotic free-rank of the modules $F_*^e R$. More specifically, if (R, \mathfrak{m}, k) is a local ring with perfect residue field, for each $e \in \mathbb{Z}_{>0}$ we write $F_*^e R = R^{\oplus a_e(R)} \oplus M_e$, where M_e has no free summands. It is easy to see that, under these assumptions, the integers $a_e(R)$ do not depend on the chosen direct sum decomposition. The *F-splitting ratio* of R is defined as

$$r_F(R) = \lim_{e \rightarrow \infty} \frac{a_e(R)}{p^{e \operatorname{sdim}(R)}},$$

where $\operatorname{sdim}(R)$ is the splitting dimension of R (see [AE05], and Section 2). The F-splitting ratio is always positive for F-pure rings, and its existence as a limit was first proved by Tucker for local rings [Tuc12], while its positivity for F-pure local rings was established in [BST12].

Observe that $r_F(R)$ is defined similarly to the F-signature of R ; in fact, it coincides with it if and only if $\operatorname{sdim}(R) = \dim(R)$. However, $r_F(R)$ is always positive for an F-pure ring, while $s(R)$ is non-zero only for strongly F-regular rings.

The splitting dimension and splitting numbers can naturally be reinterpreted for a finitely generated module M over an F-finite ring which is not necessarily local (see Section 4). We call the extension of the splitting dimension the *splitting rate* of M , and we denote it by $\operatorname{sr}(M)$. When $\operatorname{sr}(M) \geq 0$, the F-splitting ratio of M is defined as

$$r_F(M) = \lim_{e \rightarrow \infty} \frac{a_e(M)}{p^{e \operatorname{sr}(M)}},$$

provided the limit exists.

Our first main result provides strong uniform bounds on the splitting numbers of a module. An immediate consequence of this is the existence of the F-splitting ratio of a module in the local case.

Polstra was supported in part by NSF Postdoctoral Research Fellowship DMS #1703856.

Theorem A (see Theorem 4.6). *Let R be an F -finite ring, and M be a finitely generated R -module. There exists a constant C such that for all $Q \in \text{Spec}(R)$ and $e \in \mathbb{Z}_{>0}$,*

$$\left| a_e(M_Q) - p^{e \text{sr}(M_Q)} r_F(M_Q) \right| \leq C p^{e(\text{sr}(M_Q)-1)}.$$

In particular, if (R, \mathfrak{m}, k) is local and M is a finitely generated R -module, then $r_F(M)$ exists as a limit.

We provide an example to show that, in general, lower semi-continuity may not hold on the whole spectrum of a ring, even in the case when the ring is a domain (Example 5.4). This is in contrast with the behavior of several other invariants: Hilbert-Kunz multiplicity and F -signature [Pol18], Frobenius Betti numbers and Frobenius Euler characteristics [DSPY19a].

Using Theorem A and some partial lower semi-continuity results for the F -splitting ratio (see Theorem 5.1), we prove the existence of the F -splitting ratio of a module over a ring which is not necessarily local. We also relate both the splitting rate and the F -splitting ratio of a module to the respective invariants in the localizations at prime ideals. This last fact allows us to relate the positivity of $r_F(R)$ to the F -purity of R .

Theorem B (see Theorem 5.6). *Let R be an F -finite domain of prime characteristic $p > 0$. Then*

- (1) *The limit $r_F(R)$ exists.*
- (2) *We have equalities*

$$\text{sr}(R) = \min\{\text{sr}(R_P) \mid P \in \text{Spec}(R)\}$$

and

$$r_F(R) = \min\{r_F(R_P) \mid \text{sr}(R) = \text{sr}(R_P)\}.$$

- (3) *$r_F(R) > 0$ if and only if R is F -pure.*

Theorem B is here stated only for global F -splitting ratio of the ring R , under the additional assumption that it is a domain. We refer the reader to 5 for more general results on finitely generated R -modules.

Among other properties of F -splitting ratios, we prove that if R is a positively graded algebra over a local ring (R_0, \mathfrak{m}_0) , then $\text{sr}(R) = \text{sr}(R_{\mathfrak{m}})$ and $r_F(R) = r_F(R_{\mathfrak{m}})$, where $\mathfrak{m} = \mathfrak{m}_0 + R_{>0}$ (see Proposition 5.7). This result gives an analogous statement for the global F -signature (see Corollary 5.8).

In the final section of this article, we positively answer [DSPY19b, Question 4.24]. In the local case, it was proved in [BST12] that the F -signature of a Cartier algebra \mathcal{D} on R is positive if and only if the pair (R, \mathcal{D}) is strongly F -regular. These authors were able to recover the same result in the global setting, provided the Cartier algebra \mathcal{D} satisfies certain additional assumptions [DSPY19b, Theorem 2.24]. We are able to remove these extra conditions:

Theorem C. *Let R be an F -finite domain, and \mathcal{D} be a Cartier algebra on R . Then (R, \mathcal{D}) is strongly F -regular if and only if $s(R, \mathcal{D}) > 0$.*

2. BACKGROUND ON F -SPLITTING RATIO OF LOCAL RINGS

Let (R, \mathfrak{m}, k) be an F -finite local ring of prime characteristic $p > 0$. Aberbach and Enescu introduced the concepts of splitting prime and F -splitting ratio of a local F -finite ring in

[AE05]. Assume that R is F-pure, that is, the Frobenius map is pure as a map of rings. In our assumptions, this is the same as requiring that R is F-split [HR76, Corollary 5.3]. For a finitely generated R -module M , we let $\text{frk}_R(M)$ be the maximal rank of a free summand of M . Equivalently, $\text{frk}_R(M)$ is the maximal rank of a free module G for which there is a surjection $M \rightarrow G \rightarrow 0$. For all $e \in \mathbb{Z}_{>0}$, we let $a_e(R) = \text{frk}_R(F_*^e R)$ be the e -th *splitting number* of R . The *splitting dimension* of R is

$$\text{sdim}(R) := \sup \left\{ \ell \in \mathbb{Z}_{\geq 0} \mid \liminf_{e \rightarrow \infty} \frac{a_e(R)}{p^{e(\ell + \alpha(\mathfrak{m})})} > 0 \right\}.$$

The *F-splitting ratio* of R is defined to be the limit

$$r_F(R) := \lim_{e \rightarrow \infty} \frac{a_e(R)}{p^{e(\text{sdim}(R) + \alpha(\mathfrak{m}))}},$$

which always exists [Tuc12, Theorem 4.9] and is always positive for F-pure rings by work of Blickle, Schwede, and Tucker [BST12, Corollary 4.3].

Remark 2.1. Observe that, when $\text{sdim}(R) = \dim(R)$, the F-splitting ratio is equal to the F-signature of R .

Continue to let (R, \mathfrak{m}, k) denote an F-finite and F-pure local ring of prime characteristic $p > 0$. For each $e \in \mathbb{Z}_{>0}$ let $I_e = \{r \in R \mid R \xrightarrow{F_*^e(r)} F_*^e R \text{ is not pure}\}$ be the e -th splitting ideal of R . Aberbach and Enescu show in [AE05] that $\mathcal{P} := \bigcap_{e \in \mathbb{Z}_{>0}} I_e$ is a prime ideal of R and R/\mathcal{P} is a strongly F-regular local ring. The ideal \mathcal{P} is called the splitting prime of the local ring R . Moreover, it is shown in [BST12] that the splitting dimension of R is the Krull dimension of the local ring R/\mathcal{P} .

We recall that a graded \mathbb{F}_p -subalgebra \mathcal{D} of $\bigoplus_{e \in \mathbb{Z}_{\geq 0}} \text{Hom}_R(F_*^e R, R)$, with $\mathcal{D}_0 = \text{Hom}_R(R, R)$ and multiplication $\varphi \bullet \psi = \varphi \circ F_*^e \psi \in \mathcal{D}_{e+e'}$ for all $\varphi \in \mathcal{D}_e$ and $\psi \in \mathcal{D}_{e'}$, is called a Cartier algebra. If $\mathcal{D}_e = \text{Hom}_R(F_*^e R, R)$ for all e , we refer to \mathcal{D} as the full Cartier algebra on R . See [BST12] for more details on Cartier algebras.

If $I \subseteq R$ is an ideal, then we let $\mathcal{D}_{R/I}$ be the Cartier algebra on R/I whose e -th graded component is denoted by $\mathcal{D}_{R/I, e}$ and consists of R/I -linear maps $\varphi : F_*^e(R/I) \rightarrow R/I$ which can be factored through an R -linear map $\phi : F_*^e R \rightarrow R$. That is, there exists commutative diagram of R -modules of the form

$$\begin{array}{ccc} F_*^e(R/I) & \xrightarrow{\varphi} & R/I \\ \uparrow & & \uparrow \\ F_*^e R & \xrightarrow{\exists \phi} & R \end{array}$$

Observe that the construction of this Cartier algebra did not require R to be local. Moreover, if P is a prime ideal of R which contains I , then the localized Cartier algebra $(\mathcal{D}_{R/I})_P$ agrees with \mathcal{D}_{R_P/IR_P} .

We now recall the definition of splitting numbers of a pair (R, \mathcal{D}) in the local case. Let (R, \mathfrak{m}, k) be a local F-finite and F-pure ring of prime characteristic $p > 0$, and \mathcal{D} be a Cartier algebra. We let $a_e(R, \mathcal{D})$ be the largest rank of a free \mathcal{D} -summand of $F_*^e R$. More explicitly, we look at the largest rank of a free R -module $G \cong \bigoplus R$ for which there is a surjection $F_*^e R \xrightarrow{\varphi} G \rightarrow 0$, with φ that is a direct sum of elements in \mathcal{D}_e when viewed as an element of

$\mathrm{Hom}_R(F_*^e R, G) \cong \bigoplus \mathrm{Hom}_R(F_*^e R, R)$. It was proved in [BST12] that, if \mathcal{D} is the full Cartier algebra on R , and \mathcal{P} is the splitting prime of R , one has

$$a_e(R) = a_e(R/\mathcal{P}, \mathcal{D}_{R/\mathcal{P}}).$$

We record the following theorem of Blickle, Schwede, and Tucker for future reference.

Theorem 2.2 ([BST12]). *Let (R, \mathfrak{m}, k) be a local F -finite and F -pure ring of prime characteristic $p > 0$. Let \mathcal{D} be the full Cartier algebra on R , and \mathcal{P} be the splitting prime of R . Then $a_e(R) = a_e(R/\mathcal{P}, \mathcal{D}_{R/\mathcal{P}})$ for all $e \in \mathbb{Z}_{>0}$, and thus $r_F(R) = s(R/\mathcal{P}, \mathcal{D}_{R/\mathcal{P}}) = r_F(R/\mathcal{P}, \mathcal{D}_{R/\mathcal{P}})$. In particular, the F -splitting ratio of R is strictly positive.*

3. UNIFORM BOUNDS FOR SPLITTING NUMBERS

With the goal in mind of extending the theory of F -splitting ratios to modules over rings which are not necessarily local, we must first discuss and understand properties of centers of F -purity, i.e., compatibly split subvarieties, whose properties are developed by Schwede in [Sch09] and [Sch10], and by Kumar and Mehta in [KM09].

Let R be an F -finite ring of prime characteristic $p > 0$, not necessarily local, and M be a finitely generated R -module. For $e \in \mathbb{Z}_{>0}$ we let $a_e(M) = \mathrm{frk}_R(F_*^e M)$, and assume that $a_e(M) > 0$ for some e . Under these assumptions, we make the following definition.

Definition 3.1. We define the F -splitting rate of M to be

$$\mathrm{sr}(M) := \sup \left\{ \ell \in \mathbb{Z}_{\geq 0} \mid \liminf_{e \rightarrow \infty} \frac{a_e(M)}{p^{e\ell}} > 0 \right\}.$$

If (R, \mathfrak{m}, k) is local, then $\mathrm{sr}(R) = \mathrm{sdim}(R) + \alpha(\mathfrak{m})$. Moreover, if \mathcal{P} the splitting prime of (R, \mathfrak{m}, k) , then $\mathrm{sr}(R) = \gamma(R/\mathcal{P})$. When $a_e(M) = 0$ for all $e \in \mathbb{Z}_{>0}$ we set $\mathrm{sr}(M) = -1$.

Now assume that R is F -finite and F -pure, that is, $a_e(R) > 0$ for some (equivalently, for all) $e \in \mathbb{Z}_{>0}$. An ideal $P \in \mathrm{Spec}(R)$ is called a center of F -purity if for every $x \in P$ and every $e \in \mathbb{Z}_{>0}$ the map

$$R_P \xrightarrow{\cdot F_*^e x} F_*^e(R_P)$$

is not pure as a map of R_P -modules. If R is local and \mathcal{P} the splitting prime of R then \mathcal{P} is the unique maximal center of F -purity of R , [Sch10, Remark 4.4]. An important property enjoyed by all F -finite F -pure rings is that they only admit finitely many centers of F -purity [Sch09, Theorem C].

Also crucial to our proof of existence of global F -splitting ratio will be that Cartier algebras of the form $\mathcal{D}_{R/I}$ described above satisfy the following technical condition.

Condition 3.2. Let R be an F -finite ring and \mathcal{D} a Cartier algebra. We say that \mathcal{D} satisfies condition $(*)$ if we require that for each $\varphi \in \mathcal{D}_{e+1}$ that the natural map $i \circ \varphi \in \mathcal{D}_e$ where $i : F_*^e R \rightarrow F_*^{e+1} R$ is the Frobenius.

Lemma 3.3. *Let R be an F -finite ring of prime characteristic $p > 0$ and $I \subseteq R$ be an ideal. Assume that the Cartier algebra \mathcal{D} on R satisfies $(*)$. Then the Cartier algebra $\mathcal{D}_{R/I}$ on R/I satisfies condition $(*)$ as well.*

Proof. Let $\varphi \in \mathcal{D}_{R/I, e+1}$, and $i : F_*^e(R/I) \rightarrow F_*^{e+1}(R/I)$ be the Frobenius map on $F_*^e(R/I)$. We are assuming there exists a commutative diagram of R -modules of the form

$$\begin{array}{ccccc} F_*^e(R/I) & \xrightarrow{i} & F_*^{e+1}(R/I) & \xrightarrow{\varphi} & R/I \\ \uparrow & & \uparrow & & \uparrow \\ F_*^e R & \dashrightarrow & F_*^{e+1} R & \xrightarrow{\phi} & R \end{array}$$

The Frobenius map on $F_*^e(R/I)$ can be lifted by the Frobenius map on $F_*^e R$. Therefore the above commutative diagram can be filled in, and it follows that $\varphi \circ i \in \mathcal{D}_{R/I, e}$. \square

We are almost ready to prove a uniform bound result for the localized splitting numbers $a_e(R_P)$ of an F-finite ring R , but first we recall a uniform bound result of [Pol18]. We use the following notation: given a prime $P \in \text{Spec}(R)$ we let $\alpha(P) = \log_p[F_*\kappa(P) : \kappa(P)]$ and $\gamma(R) = \max\{\alpha(P) \mid P \in \text{Spec}(R)\}$. Moreover, given a pair (R, \mathcal{D}) , $P \in \text{Spec}(R)$ and $e \in \mathbb{Z}_{>0}$, we let $a_e(R_P, \mathcal{D}_P)$ be the maximal rank of a free \mathcal{D}_P -summand of $F_*^e(R_P)$. In the case when $\mathcal{D} = \text{Hom}_R(F_*^e R, R)$ is the full Cartier algebra, we simply write $a_e(R_P)$, which is also equal to $\text{frk}_{R_P}(F_*^e R_P)$.

Theorem 3.4 ([Pol18]). *Let R be an F-finite ring, and \mathcal{D} be a Cartier algebra satisfying condition (*). There exists a constant C such that for all $P \in \text{Spec}(R)$ and all $e \in \mathbb{Z}_{>0}$*

$$\left| a_e(R_P, \mathcal{D}_P) - p^{e\gamma(R_P)} s(R_P, \mathcal{D}_P) \right| \leq Cp^{e(\gamma(R)-1)}.$$

Using this, we obtain uniform bounds for the difference of localized splitting numbers of an F-finite F-pure ring and the corresponding F-splitting ratios.

Theorem 3.5. *Let R be an F-finite ring and F-pure ring. There is a constant $C \in \mathbb{R}$ such that for all $P \in \text{Spec}(R)$ and $e \in \mathbb{Z}_{>0}$*

$$\left| a_e(R_P) - p^{e \text{sr}(R_P)} r_F(R_P) \right| \leq Cp^{e(\text{sr}(R_P)-1)}.$$

Proof. Let $Y = \{\mathfrak{p}_1, \dots, \mathfrak{p}_N\}$ be the finitely many centers of F-purity of $\text{Spec}(R)$, and \mathcal{D} be the full Cartier algebra on R . Observe that \mathcal{D} trivially satisfies condition (*). For each \mathfrak{p}_i , let C_i be a constant as in Theorem 3.4 for the pair $(R/\mathfrak{p}_i, \mathcal{D}_{R/\mathfrak{p}_i})$. We claim that we can choose $C = \max\{C_1, \dots, C_N\}$. In fact, given $P \in \text{Spec}(R)$, there is a unique $\mathfrak{p}_i \in Y$ such that $\mathfrak{p}_i R_P$ is the splitting prime of R_P . If we let $S = R/\mathfrak{p}_i$, by Theorem 2.2 we have that $a_e(R_P) = a_e(S_P, \mathcal{D}_{S_P})$ and $r_F(R_P) = r_F(S_P, \mathcal{D}_{S_P})$. As the Cartier algebra \mathcal{D}_S still satisfies condition (*), it then follows from Theorem 3.4 that

$$\begin{aligned} \left| a_e(R_P) - p^{e \text{sr}(R_P)} r_F(R_P) \right| &= \left| a_e(S_P, \mathcal{D}_{S_P}) - p^{e \text{sr}(R_P)} r_F(S_P, \mathcal{D}_{S_P}) \right| \\ &\leq C_i p^{e(\gamma(S_P)-1)} \leq Cp^{e(\text{sr}(R_P)-1)}. \end{aligned} \quad \square$$

A consequence of Theorem 3.5 is the following:

Corollary 3.6. *Let R be an F-finite and F-pure ring of prime characteristic $p > 0$. Then the normalized splitting number functions $\tilde{a}_e : \text{Spec}(R) \rightarrow \mathbb{R}$ mapping $P \mapsto a_e(R_P)/p^{e \text{sr}(R_P)}$ converge uniformly as $e \rightarrow \infty$ to the F-splitting ratio function $r_F : \text{Spec}(R) \rightarrow \mathbb{R}$ mapping $P \mapsto r_F(R_P)$.*

4. F-SPLITTING RATIO OF MODULES OVER LOCAL RINGS

The theory of splitting ratios over a local ring developed in [AE05] and [BST12] only concerns itself with the Frobenius splitting numbers $a_e(R)$ of a local ring (R, \mathfrak{m}, k) . In this section we extend the local theory by studying the Frobenius splitting numbers of finitely generated modules. We first make a more general definition.

Definition 4.1. Let R be an F-finite ring of prime characteristic $p > 0$, and M be a finitely generated R -module. If $a_e(M) > 0$ for some $e \in \mathbb{Z}_{>0}$, we let

$$r_F(M) = \lim_{e \rightarrow \infty} \frac{a_e(M)}{p^{e \operatorname{sr}(M)}},$$

provided the limit exists. If $a_e(M) = 0$ for all $e \in \mathbb{Z}_{>0}$, we let $r_F(M) = 0$.

The goal of this section is to prove the existence of the limit when R is assumed to be local.

We begin with the following observation.

Lemma 4.2. *Let (R, \mathfrak{m}, k) be a local F-finite ring of prime characteristic $p > 0$ and let M be a finitely generated R -module. If $a_{e_0}(M) > 0$ for some $e_0 \in \mathbb{Z}_{>0}$ then $\operatorname{sr}(M) = \operatorname{sr}(R)$.*

Proof. Choose an onto R -linear map $R^{\oplus n} \rightarrow M$. Then $a_e(M) \leq n a_e(R)$ and it follows that $\operatorname{sr}(M) \leq \operatorname{sr}(R)$. If $F_*^{e_0} M \cong R \oplus M_{e_0}$ for some e_0 then $F_*^{e+e_0} M \cong F_*^e R \oplus F_*^e M_{e_0}$ for each $e \in \mathbb{Z}_{>0}$. Therefore $a_e(R) \leq a_{e+e_0}(M)$ for each $e \in \mathbb{Z}_{>0}$ and $\operatorname{sr}(R) \leq \operatorname{sr}(M)$. \square

In what follows, it will be useful to keep track of the primes P for which the splitting rate of M is non-negative. We make the following definition.

Definition 4.3. Let R be an F-finite ring and M a finitely generated R -module. The *F-split locus* of M is $\operatorname{fs}(M) = \{P \in \operatorname{Spec}(R) \mid F_*^e(M_P) \text{ has a free summand for some } e > 0\}$.

Observe that, if $F_*^e(M_P)$ has a free summand, then so does $F_*^e(R_P)$. Therefore $\operatorname{fs}(M) \subseteq \operatorname{fs}(R)$. Moreover, Lemma 4.2 proves that, if $P \in \operatorname{fs}(M)$, then the splitting rates of M_P and R_P agree. Our next lemma establishes the existence of the F-splitting ratio of a finitely generated module over a local ring (R, \mathfrak{m}, k) under the assumption that \mathfrak{m} is the splitting prime ideal of R .

Lemma 4.4. *Let (R, \mathfrak{m}, k) be an F-finite and F-pure local ring, with \mathfrak{m} being its splitting prime. Let $\gamma = \gamma(\mathfrak{m})$. For every $e \geq 0$, write $F_*^e M \cong R^{\oplus a_e(M)} \oplus M_e$. Then*

- (1) *The sequence $\{a_e(R)/p^{e\gamma}\}$ is the constant sequence $\{1\}$. In particular $r_F(R) = 1$.*
- (2) *The sequence $\{a_e(M)/p^{e\gamma}\}_{e \geq 0}$ is a bounded non-decreasing sequence of integers, and therefore eventually constant. In particular, the F-splitting ratio $r_F(M)$ exists. Moreover, $\operatorname{sr}(M) = \gamma \iff r_F(M) > 0 \iff \mathfrak{m} \in \operatorname{fs}(M)$.*
- (3) *If $a_e(M)/p^{e\gamma} = r_F(M)$ then $a_{e'}(M_e) = 0$ for all $e' \geq 0$.*

Proof. As discussed before, if we let $I_e = \{r \in R \mid R \xrightarrow{F_*^{e\gamma}} F_*^e R \text{ does not split}\}$ then I_e is an \mathfrak{m} -primary ideal such that $\lambda(R/I_e) = \frac{a_e(R)}{p^{e\gamma}}$, and $\bigcap_{e \in \mathbb{Z}_{>0}} I_e$ is the splitting prime of R . Hence $I_e = \mathfrak{m}$ for each $e \in \mathbb{Z}_{>0}$ and therefore $\lambda(R/I_e) = \frac{a_e(R)}{p^{e\gamma}} = 1$ for each $e \in \mathbb{Z}_{>0}$.

Given finitely generated module M we let $I_e(M) = \{m \in M \mid R \xrightarrow{F_*^e m} F_*^e M \text{ does not split}\}$. It is known, and easy to prove, that $I_e(M)$ is a submodule of M containing $\mathfrak{m}^{[p^e]}M$ and $\lambda(M/I_e(M)) = \frac{a_e(M)}{p^{e\gamma}}$ is an integer.

As M is a homomorphic image of $R^{\oplus n}$ for some integer $n \geq 0$, we see that

$$\frac{a_e(M)}{p^{e\gamma}} \leq \frac{a_e(R^{\oplus n})}{p^{e\gamma}} = \frac{a_e(R)n}{p^{e\gamma}} = n.$$

Also observe that for all $e' \geq 0$, we have $a_{e+e'}(M) = a_e(M)a_{e'}(R) + a_{e'}(M_e) = a_e(M)p^{e'\gamma} + a_{e'}(M_e)$ and hence

$$\frac{a_{e+e'}(M)}{p^{(e+e')\gamma}} = \frac{a_e(M)}{p^{e\gamma}} + \frac{a_{e'}(M_e)}{p^{(e+e')\gamma}} \geq \frac{a_e(M)}{p^{e\gamma}}.$$

In summary, $\{a_e(M)/p^{e\gamma}\}_{e \geq 0}$ is a non-decreasing sequence of integer values with an upper bound. So it is eventually constant. All remaining claims follow immediately. \square

Let (R, \mathfrak{m}, k) be a local ring, not necessarily of prime characteristic, and M a finitely generated R -module. Similar to the above, we define $I(M) = \{m \in M \mid R \xrightarrow{m} M \text{ does not split}\}$. Then $I(M) \subseteq M$ is a submodule of M satisfying $\mathfrak{m}M \subseteq I(M)$ and $\lambda(M/I(M)) = \text{frk}(M)$. We refer to $I(M)$ as the non-split submodule of M . If (R, \mathfrak{m}, k) is F-finite then $I(F_*^e M) = F_*^e I_e(M)$. Our next lemma studies the behavior of non-split submodules under R -linear maps.

Lemma 4.5. *Let (R, \mathfrak{m}, k) be a local ring (of any characteristic), let M, N and K be finitely generated R -modules, $f \in \text{Hom}_R(M, N)$ and $g \in \text{Hom}_R(N, K)$. Let $I(M), I(N)$ and $I(K)$ be the non-split submodules of M, N and K respectively.*

- (1) *We have $\text{frk}(N) \geq \lambda(M/(g \circ f)^{-1}(I(K)))$.*
- (2) *Further assume that R is an F-finite ring of prime characteristic p , $M = K$ and $g \circ f = c1_M$ for some $c \in R$. Then, for all $e \geq 0$,*

$$a_e(N) \geq a_e(M) - \lambda(M/(I_e(M) + cM))p^{e\gamma(\mathfrak{m})}.$$

Proof. For (1) first observe that $g(I(N)) \subseteq I(K)$. Else, if there exists $n \in I(N)$ such that $g(n) \notin I(K)$ then there is $\varphi : K \rightarrow R$ such that $\varphi(g(n)) = 1$ contradicting the assumption $n \in I(N)$. Therefore $g(f^{-1}(I(N))) \subseteq g(I(N)) \subseteq I(K)$. In particular, $f^{-1}(I(N)) \subseteq (g \circ f)^{-1}(I(K))$ and hence

$$\text{frk}(N) = \lambda(N/I(N)) \geq \lambda(M/f^{-1}(I(N))) \geq \lambda(M/(g \circ f)^{-1}(I(K))).$$

We now prove part (2). Suppose (R, \mathfrak{m}, k) is an F-finite ring of prime characteristic $p > 0$. For each $e \geq 0$, the induced maps $F_*^e f$ and $F_*^e g$ satisfy $F_*^e g \circ F_*^e f = (F_*^e c)1_{F_*^e M}$. So $(F_*^e g \circ F_*^e f)^{-1}(I(F_*^e M)) = (I(F_*^e M) :_{F_*^e M} F_*^e c) = F_*^e(I_e(M) :_M c)$. By (1), we see

$$\begin{aligned} a_e(N) &= \text{frk}(F_*^e N) \geq \lambda(F_*^e M/F_*^e(I_e(M) :_M c)) = \lambda(M/(I_e(M) :_M c))p^{e\gamma(\mathfrak{m})} \\ &= [\lambda(M/I_e(M)) - \lambda(M/(I_e(M) + cM))]p^{e\gamma(\mathfrak{m})} \\ &= \lambda(M/I_e(M))p^{e\gamma(\mathfrak{m})} - \lambda(M/(I_e(M) + cM))p^{e\gamma(\mathfrak{m})} \\ &= a_e(M) - \lambda(M/(I_e(M) + cM))p^{e\gamma(\mathfrak{m})}. \end{aligned}$$

The equation $\lambda(M/(I_e(M) :_M c)) = \lambda(M/I_e(M)) - \lambda(M/(I_e(M) + cM))$ follows since length is additive and there is short exact sequence

$$0 \rightarrow M/(I_e(M) :_M c) \rightarrow M/I_e(M) \rightarrow M/(I_e(M) + cM) \rightarrow 0. \quad \square$$

We are now ready to accomplish two tasks simultaneously: proving the existence of the F-splitting ratio of a finitely generated module over a local ring, and a uniform convergence result which extends Theorem 3.5 to finitely generated modules.

Theorem 4.6. *Let R be an F-finite ring, M a finitely generated R -module, and for each prime ideal $Q \in \text{fs}(R)$ let $\mathcal{P}(Q)$ be the splitting prime ideal of R_Q . Then $r_F(M_Q) = r_F(M_{\mathcal{P}(Q)})r_F(R_Q)$ and $\text{sr}(M_Q) = \text{sr}(M_{\mathcal{P}(Q)})$ for all $Q \in \text{fs}(R)$. Moreover, there exists a constant C such that for all $Q \in \text{Spec}(R)$ and $e \in \mathbb{Z}_{>0}$,*

$$\left| a_e(M_Q) - p^{e \cdot \text{sr}(M_Q)} r_F(M_Q) \right| \leq C p^{e(\text{sr}(M_Q)-1)}.$$

Proof. If $Q \notin \text{fs}(R)$ then $a_e(M_Q) = a_e(R_Q) = 0$ for all $e \in \mathbb{Z}_{>0}$ and any choice of constant $C \geq 0$ satisfies the desired inequality for all such prime ideals. Furthermore, the F-pure locus $\text{fs}(R)$ is open in $\text{Spec}(R)$ and is covered by finitely many principal open sets of the form $\text{Spec}(R_f)$ with each R_f being F-pure. Thus we may prove the theorem for each of these pieces of the affine cover and assume for the remainder of the proof that R is an F-pure ring. In particular, R has only finitely many centers of F-purity (see [Sch09, Theorem C] and [KM09, Theorem 1.1]).

For each center of F-purity \mathcal{P} , let $\mathcal{Q}(\mathcal{P}) = \{Q \in \text{Spec}(R) \mid \mathcal{P}(Q) = \mathcal{P}\}$. If $Q \in \text{Spec}(R)$ then $\mathcal{P}(Q) = \mathcal{P}$ if and only if $\mathcal{P}R_Q$ is the splitting prime ideal, i.e. the maximal center of F-purity, of R_Q . Thus if $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ are the centers of F-purity containing, but not equal to \mathcal{P} , and if $s \in (\mathcal{P}_1 \cap \dots \cap \mathcal{P}_\ell) \setminus \mathcal{P}$, then $\mathcal{Q}(\mathcal{P}) = V(\mathcal{P}) \cap D(s)$. Here, $D(s)$ denotes the open subset of $\text{Spec}(R)$ consisting of those prime ideals that do not contain s . Since $\text{Spec}(R)$ can be covered by finitely many sets of this form it is enough to show the existence of a uniform constant C for which the desired inequality holds for each of the primes in the set $\mathcal{Q}(\mathcal{P})$.

If $r_F(M_{\mathcal{P}}) = 0$ then $r_F(M_Q) = 0$ and the conclusion holds for all $Q \in \mathcal{Q}(\mathcal{P})$. So we assume $r_F(M_{\mathcal{P}}) > 0$ for the rest of proof. Let $\gamma = \gamma(\mathcal{P}) = \text{sr}(M_{\mathcal{P}})$. By Lemma 4.4, there exists e_0 such that $a_{e_0}(M_{\mathcal{P}})/p^{e_0\gamma} = r_F(M_{\mathcal{P}})$. Let $a = a_{e_0}(M_{\mathcal{P}})$. Then $F_*^{e_0} M_{\mathcal{P}} \cong R_{\mathcal{P}}^{\oplus a} \oplus (M_{e_0})_{\mathcal{P}}$ over $R_{\mathcal{P}}$, for some finitely generated R -module M_{e_0} . Lifting to R , we obtain R -linear maps

$$R^{\oplus a} \rightarrow F_*^{e_0} M \rightarrow R^{\oplus a} \quad \text{and} \quad F_*^{e_0} M \rightarrow R^{\oplus a} \oplus M_{e_0} \rightarrow F_*^{e_0} M$$

such that both compositions are multiplication by some $c \in R \setminus \mathcal{P}$. Applying Lemma 4.5 to the composition map $R^{\oplus a} \rightarrow F_*^{e_0} M \rightarrow R^{\oplus a}$, we see that for all $Q \in \mathcal{Q}(\mathcal{P})$ and $e \geq 0$,

$$a_{e_0+e}(M_Q) \geq a \cdot (a_e(R_Q) - \lambda(R_Q/(I_e(R_Q) + cR_Q))) p^{e\gamma(Q)}.$$

Therefore

$$\begin{aligned} \frac{a_{e_0+e}(M_Q)}{p^{(e_0+e)\gamma}} &\geq \frac{a(a_e(R_Q) - \lambda(R_Q/(I_e(R_Q) + cR_Q))) q^{\gamma(Q)}}{p^{(e_0+e)\gamma}} \\ &= \frac{a}{p^{e_0\gamma}} \left(\frac{a_e(R_Q)}{p^{e\gamma}} - \frac{\lambda(R_Q/(I_e(R_Q) + cR_Q))}{p^{e \dim(R_Q/\mathcal{P}R_Q)}} \right) \\ &\geq r_F(M_{\mathcal{P}}) \left(\frac{a_e(R_Q)}{p^{e\gamma}} - \frac{\lambda(R_Q/(Q^{[p^e]} + \mathcal{P} + cR)_Q)}{p^{e \dim(R_Q/\mathcal{P}R_Q)}} \right). \end{aligned}$$

By Theorem 3.5 there exists a constant C_1 , independent of e and $Q \in \mathcal{Q}(\mathcal{P})$, such that $\frac{a_e(R_Q)}{p^{e\gamma}} \geq r_F(R_Q) - \frac{C_1}{p^e}$, where $\gamma = \text{sr}(M_{\mathcal{P}})$ as above. This is because, by Lemma 4.2, we have $\text{sr}(M_{\mathcal{P}}) = \text{sr}(R_{\mathcal{P}})$. Moreover, since $\text{sdim}(R_Q) = \dim(R_Q/\mathcal{P}R_Q)$, we have $\text{sr}(R_{\mathcal{P}}) = \text{sr}(R_Q)$ for all $Q \in \mathcal{Q}(\mathcal{P})$. Thus, $\gamma = \text{sr}(M_{\mathcal{P}}) = \text{sr}(R_{\mathcal{P}}) = \text{sr}(R_Q)$ for all $Q \in \mathcal{Q}(\mathcal{P})$. By [Pol18, Proposition 3.3], there exists a constant C_2 , independent of e and $Q \in \mathcal{Q}(\mathcal{P})$, such that $\frac{\lambda(R_Q/(Q^{[p^e]} + \mathcal{P} + cR)_Q)}{p^{e \dim(R_Q/\mathcal{P}R_Q)}} \leq \frac{C_2}{p^e}$. Therefore the constant $C = C_1 + r_F(M_{\mathcal{P}})C_2p^{e_0}$, which is independent of e and $Q \in \mathcal{Q}(\mathcal{P})$, is such that

$$\frac{a_{e_0+e}(M_Q)}{p^{(e_0+e)\gamma}} \geq r_F(M_{\mathcal{P}})r_F(R_Q) - \frac{C}{p^{e+e_0}}.$$

An argument similar to the above, applied to the composition of maps $F_*^{e_0}M \rightarrow R^{\oplus a} \oplus M_{e_0} \rightarrow F_*^{e_0}M$, will provide the existence of a constant C' , independent of $Q \in \mathcal{Q}(\mathcal{P})$ and e , such that

$$\frac{a_{e_0+e}(M_Q)}{p^{(e_0+e)\gamma}} \leq r_F(M_{\mathcal{P}})r_F(R_Q) + \frac{C'}{p^{e+e_0}}.$$

This shows, in particular, that $\frac{a_e(M_Q)}{p^{e\gamma}}$ converges uniformly to $r_F(M_{\mathcal{P}})r_F(R_Q) > 0$, and thus $\gamma = \text{sr}(M_{\mathcal{P}}) = \text{sr}(M_Q)$. All assertions of the theorem now follow. \square

5. LOWER SEMI-CONTINUITY AND GLOBAL F-SPLITTING RATIO

The main purpose of this section is to prove existence of the global F-splitting ratio of a finitely generated module M .

We first need to establish some lower semi-continuity results for F-splitting ratios. We will see that, unlike the F-signature, the F-splitting ratio of an F-finite domain may not be a lower semi-continuous function.

Let R be an F-finite ring and M a finitely generated R -module. For each $-1 \leq \ell \leq \gamma(R)$ we set $W_{\ell}(M) = \{P \in \text{Spec}(R) \mid \text{sr}(M_P) = \ell\}$. From Lemma 4.2 and subsequent observations, we have that $W_{\ell}(M) = W_{\ell}(R) \cap \text{fs}(M)$ for all $\ell \geq 0$.

Theorem 5.1. *Let R be an F-finite and F-pure ring of prime characteristic $p > 0$, set $X = \text{Spec}(R)$ and let M be a finitely generated R -module. Then there is a finite stratification of X into locally closed quasi-compact subsets such that the restriction of the F-splitting ratio function on each subset is lower semi-continuous. Specifically, $X = \bigcup_{i=-1}^{\gamma(R)} W_i(M)$, $W_i(M) \cap W_j(M) = \emptyset$ whenever $i \neq j$, the sets $W_i(M)$ are locally closed and quasi-compact, and the function $r_F : \text{Spec}(R) \rightarrow \mathbb{R}$ mapping $P \mapsto r_F(M_P)$ is lower semi-continuous when restricted to each $W_i(M)$.*

Proof. The functions $a_e : \text{Spec}(R) \rightarrow \mathbb{R}$ mapping $P \mapsto a_e(M_P)$ are easily checked to be lower semi-continuous, see [EY11, Proposition 2.2]. The normalized functions \tilde{a}_e mapping $P \mapsto a_e(M_P)/p^{e \text{sr}(M_P)}$ are therefore lower semi-continuous when restricted to each of the subsets $W_{\ell}(M)$. It follows that the function r_F is lower semi-continuous when restricted to each $W_{\ell}(M)$ as it is realized as the uniform limit of lower semi-continuous functions by Theorem 4.6. It is also easy to see that the sets $W_{-1}(M), W_0(M), \dots, W_{\gamma(R)}(M)$ are disjoint and $X = \bigcup_{i=-1}^{\gamma(R)} W_i(M)$. It remains to show each of the sets $W_{\ell}(M)$ are locally closed and quasi-compact.

We adopt the convention that $W_i(M) = \emptyset$ if $i < -1$, and we let $\mathcal{P}(Q)$ denote the splitting prime of R_Q . For every $Q \in \text{Spec}(R)$, and every $-1 \leq \ell \leq \gamma(R)$, Theorem 4.6 shows that $\text{sr}(M_Q) = \text{sr}(M_{\mathcal{P}(Q)})$, and hence $Q \in W_\ell(M)$ if and only if $\mathcal{P}(Q) \in W_\ell(M)$.

Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_N\}$ be the finitely many centers of F-purity of R that are contained in $\text{fs}(M)$. Relabeling if necessary, we may assume $\gamma(R/\mathfrak{p}_j) = \ell$ if and only if $1 \leq j \leq i$, and $\gamma(R/\mathfrak{p}_j) < \ell$ if and only if $i+1 \leq j \leq t$. Observe that

$$W_\ell(M) = \left(\bigcup_{j=1}^i V(\mathfrak{p}_j) \right) \setminus \left(\bigcup_{j=i+1}^t V(\mathfrak{p}_j) \right) = \left(\bigcup_{j=1}^i V(\mathfrak{p}_j) \right) \cap \left(X \setminus \bigcup_{j=i+1}^t V(\mathfrak{p}_j) \right),$$

hence it is a locally closed set. Finally, note that every locally closed set of $\text{Spec}(R)$, with R Noetherian, is quasi-compact. \square

Corollary 5.2. *Let R be an F-finite and F-pure ring and let M be a finitely generated R -module. For $\ell \geq 0$, if $W_\ell(M) \neq \emptyset$, then the F-splitting ratio function defined by $r_F : \text{Spec}(R) \rightarrow \mathbb{R}$ mapping $P \mapsto r_F(M_P)$ has a nonzero minimum value when restricted to $W_\ell(M)$.*

Proof. The function r_F is lower semi-continuous when restricted to the non-empty quasi-compact set $W_\ell(M)$ and therefore attains a minimum value. \square

The F-splitting ratio function is generally not a lower semi-continuous function when viewed as a function on the spectrum of a ring. We provide an example of such a ring, but first we need a lemma.

Lemma 5.3. *Let (R, \mathfrak{m}, k) be an F-finite and F-pure ring satisfying the following:*

- (1) R is F-pure;
- (2) R is not strongly F-regular;
- (3) R_P is strongly F-regular for all $P \neq \mathfrak{m}$;
- (4) R_P is not regular for some $P \neq \mathfrak{m}$.

Then the F-splitting ratio function $r_F : \text{Spec}(R) \rightarrow \mathbb{R}$ is not lower semi-continuous.

Proof. For each $e \in \mathbb{Z}_{>0}$ let $I_e = \{r \in R \mid R \xrightarrow{F_*^e(r)} F_*^e R \text{ is not pure}\}$ be the e th splitting ideal of R and set $\mathcal{P} = \bigcap_{e \in \mathbb{Z}_{>0}} I_e$. Recall that \mathcal{P} is referred to as the splitting prime of R , and since R is assumed to be not strongly F-regular, the closed set $V(\mathcal{P})$ is contained in the non-strongly F-regular locus of R . Therefore $\mathcal{P} = \mathfrak{m}$ and it is straightforward to check that $a_e(R) = a_e(R/\mathfrak{m}) = [k^{1/p^e} : k]$ for all e . In particular, $a_e(R)/p^{e \text{sr}(R)} = 1$ for all e and therefore $r_F(R) = 1$. However, localizing at a prime $P \neq \mathfrak{m}$ for which R_P is not regular it follows R_P is strongly F-regular by assumption but not regular and therefore $r_F(R_P) = s(R_P) < 1$ by [HL02, Corollary 16] and therefore the F-splitting function is not lower semi-continuous. \square

Theorem 5.4. *There exist an F-finite ring R for which the F-splitting ratio function is not lower semi-continuous as a function from $\text{Spec}(R)$ to \mathbb{R} .*

Proof. Let k be a perfect field of prime characteristic p and let A be a non-regular strongly F-regular ring of finite type over k , write $A = k[x_1, \dots, x_n]/I$, with $I \subseteq (x_1, \dots, x_n)$, and assume $A_{(x_1, \dots, x_n)}$ is a non-regular local ring. Let $B = A[v]$ and $R = \{f \in B \mid f(0, \dots, 0, 0) = f(0, \dots, 0, 1)\}$ localized at the maximal ideal $R \cap (x_1, \dots, x_n, v)B = R \cap (x_1, \dots, x_n, v-1)B$. The ring R is realized as a fiber product, i.e. a gluing of the local rings $B_{(x_1, \dots, x_n, v)}$ and

$B_{(x_1, \dots, x_n, v-1)}$ at their maximal ideals. It readily follows that the conductor ideal of R inside its normalization is the unique maximal ideal of R and R is isomorphic to localizations of B on the punctured spectrum. Hence R is strongly F -regular on the punctured spectrum, but not an isolated singularity. Moreover, R is an F -pure ring since there exists splittings of $B_{(x_1, \dots, x_n, v)}$ and $B_{(x_1, \dots, x_n, v-1)}$ compatible at the residue field level of these rings. Moreover, the conductor ideal is compatible under all R -linear maps $F_*^e R \rightarrow R$ by [MS12, Lemma 3.1]. Therefore R is not strongly F -regular and satisfies all hypotheses of Lemma 5.3.¹ \square

For convenience of the reader, we recall the following:

Theorem 5.5 ([Sta82, DSPY18]). *Let R be a Noetherian ring of Krull dimension $d < \infty$ and M a finitely generated R -module. If $\text{frk}_{R_P}(M_P) \geq \dim(R/P) + k$ for all $P \in \text{Spec}(R)$, then $\text{frk}_R(M) \geq k$. In particular, $\text{frk}_R(M) \geq \min\{\text{frk}_{R_P}(M_P) \mid P \in \text{Spec}(R)\} - d$.*

We are finally ready to show the existence of the global F -splitting ratio of modules, and relates it to the F -splitting ratio of the localization at prime ideals.

Theorem 5.6. *Let R be an F -finite ring of prime characteristic $p > 0$ and M a finitely generated module. Then*

- (1) *We have $\text{sr}(M) = \min\{\text{sr}(M_P) \mid P \in \text{Spec}(R)\}$.*
- (2) *The limit $r_F(M) = \lim_{e \rightarrow \infty} \frac{a_e(M)}{p^{e \text{sr}(M)}}$ exists, and it is positive if $\text{sr}(M) \geq 0$.*
- (3) *We have $r_F(M) = \min\{r_F(M_P) \mid \text{sr}(M_P) = \text{sr}(M)\}$. In particular, $r_F(M)$ is positive whenever there exists $e \in \mathbb{Z}_{>0}$ and onto R -linear map $F_*^e M \rightarrow R$.*
- (4) *If $\text{sr}(R) = 0$, then the sequence $\{a_e(R)\}$ is the constant sequence $\{1\}$. Therefore, we have $r_F(R) = 1$.*
- (5) *If $\text{sr}(M) = 0$ then the sequence $\{a_e(M)\}$ is a non-decreasing sequence of eventually positive integers bounded from above, hence is eventually the constant sequence $\{r_F(M)\}$.*

Proof. If there exists a prime ideal $P \in \text{Spec}(R)$ such that $a_e(M_P) = 0$ for all $e \in \mathbb{Z}_{>0}$, i.e., if $W_{-1}(M) \neq \emptyset$, then all statements of the theorem trivially follow, and we have $r_F(M) = 0$.

For the remainder of the proof, we assume that $W_{-1}(M) = \emptyset$. Since $a_e(M) \leq a_e(M_P)$ for all $P \in \text{Spec}(R)$, it easily follows that $\text{sr}(M) \leq \min\{\text{sr}(M_P) \mid P \in \text{Spec}(R)\}$.

First, assume that $\min\{\text{sr}(M_P) \mid P \in \text{Spec}(R)\} > 0$. For each $e \in \mathbb{Z}_{>0}$, we let $P_e \in \text{Spec}(R)$ be such that $a_e(M_{P_e}) = \min\{a_e(M_P) \mid P \in \text{Spec}(R)\}$. If we set $d = \dim(R)$, it follows from Theorem 5.5 that $a_e(M) \geq a_e(M_{P_e}) - d$. Let C be as in Theorem 4.6, and let $r = \min\{r_F(M_P) \mid P \in \text{Spec}(R)\}$. Such an r exists, and is positive by Corollary 5.2. In particular, we have

$$\begin{aligned} a_e(M) &\geq a_e(M_{P_e}) - d \geq r_F(M_{P_e}) p^{e \text{sr}(M_{P_e})} - C p^{e(\text{sr}(M_{P_e})-1)} - d \\ &\geq r p^{e \text{sr}(M_{P_e})} - C p^{e(\text{sr}(M_{P_e})-1)} - d. \end{aligned}$$

Since $\text{sr}(M_{P_e}) > 0$, it follows that $a_e(M) \geq \frac{r p^e}{2}$ for all $e \gg 0$, and therefore $\text{sr}(M) > 0$. Moreover, we have that $\text{sr}(M_{P_e}) > \text{sr}(M)$ only for finitely many values of e . Else, from the

¹If the reader was interested in finding an example of normal ring whose F -splitting ratio function is not lower semi-continuous, then one could instead consider the cone of a singular Calabi-Yau 3-fold and show such a ring localized at the homogeneous maximal ideal satisfies the hypotheses of Lemma 5.3.

inequalities above we would get

$$r \leq \frac{a_e(M)}{p^{e \operatorname{sr}(M_{P_e})}} + \frac{C}{p^e} + \frac{d}{p^{e \operatorname{sr}(M_{P_e})}} \leq \frac{a_e(M)}{p^{e(\operatorname{sr}(M)+1)}} + \frac{C}{p^e} + \frac{d}{p^{e(\operatorname{sr}(M)+1)}},$$

for infinitely many values of e . Because $\operatorname{sr}(M) > 0$, the expression on the right hand side can be made arbitrarily close to 0 for $e \gg 0$, contradicting the fact that $r > 0$. Therefore we have $\operatorname{sr}(M_{P_e}) = \operatorname{sr}(M)$ for all $e \gg 0$ and, in particular, this gives the reverse inequality $\operatorname{sr}(M) \geq \min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\}$. This finishes the proof of (1) under the assumption that $\min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\} > 0$.

Continue to assume that $\ell = \operatorname{sr}(M) > 0$, and let $P_e \in \operatorname{Spec}(R)$ be as above. We have already observed that $\operatorname{sr}(M_{P_e}) = \ell$ for all $e \gg 0$. Moreover, there are inequalities

$$\frac{a_e(M_{P_e}) - d}{p^{e\ell}} \leq \frac{a_e(M)}{p^{e\ell}} \leq \frac{a_e(M_{P_e})}{p^{e\ell}}.$$

Under the assumption that $\ell > 0$, parts (2) and (3) follow if $\lim_{e \rightarrow \infty} \frac{a_e(M_{P_e})}{p^{e\ell}}$ exists and is equal to $\min\{r_F(M_P) \mid \operatorname{sr}(M_P) = \ell\}$. But this is indeed the case since the F-splitting ratio function restricted to the quasi-compact set $W_\ell(M) = \{P \in \operatorname{Spec}(R) \mid \operatorname{sr}(M_P) = \ell\}$ is the uniform limit of the lower semi-continuous functions $\frac{a_e(-)}{p^{e\ell}}$. In particular, the minimum the functions $\frac{a_e(-)}{p^{e\ell}}$ on $W_\ell(M)$ converges to the minimum of the F-splitting ratio functions on $W_\ell(M)$. This proves (2) and (3) under the assumption that $\min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\} > 0$.

Now we prove (4), so we assume $\operatorname{sr}(R) = 0$. By what we have shown above, we must necessarily have $0 = \operatorname{sr}(R) \leq \min\{\operatorname{sr}(R_P) \mid P \in \operatorname{Spec}(R)\} \leq 0$, and thus $\operatorname{sr}(R_P) = 0$ for some $P \in \operatorname{Spec}(R)$. Observe that we have $\operatorname{sr}(R_P) \geq \alpha(P)$, with equality if and only if PR_P is the splitting prime of R_P . Thus, $\operatorname{sr}(R_P) = 0$ implies that PR_P is the splitting prime of R_P , and that $\kappa(P)$ is perfect. It is well-known that an F -finite ring is F -pure if and only if R_P is F -pure for all $P \in \operatorname{Spec}(R)$, see for example [ST12, Exercise 2.10]. It follows from Lemma 4.4 that $1 \leq a_e(R) \leq a_e(R_P) = 1$ for all $e \in \mathbb{Z}_{>0}$, and therefore $a_e(R) = 1$ for all $e \in \mathbb{Z}_{>0}$. This proves (4).

Now suppose that $\min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\} = 0$. Let $P \in \operatorname{Spec}(R)$ be such that $\operatorname{sr}(M_P) = 0$. By Lemma 4.2, this also gives $\operatorname{sr}(R_P) = \operatorname{sr}(R) = 0$. To prove (5), we choose for each $e \in \mathbb{Z}_{>0}$ a direct sum decomposition $F_*^e M \cong R^{\oplus a_e(M)} \oplus M_e$. Then

$$F_*^{e+1} M \cong F_*^e R^{\oplus a_e(M)} \oplus F_*^e M_e.$$

As R is F -pure, $F_*^e R^{\oplus a_e(M)}$ has a free summand of rank $a_e(M)$, and therefore $a_{e+1}(M) \geq a_e(M)$. To see that the sequence $\{a_e(M)\}$ is bounded from above, choose an onto map $R^{\oplus N} \rightarrow M$. By part (4), the condition that $\operatorname{sr}(R) = 0$ implies that $a_e(R^{\oplus N}) = N$ for each $e \in \mathbb{Z}_{>0}$, and therefore $a_e(M) \leq N$ for all $e \in \mathbb{Z}_{>0}$. We have now proven, under the assumption that $\min\{\operatorname{sr}(M_P) \mid P \in \operatorname{Spec}(R)\} = 0$, that the sequence $\{a_e(M)\}$ is a non-decreasing sequence of non-negative integers, and is therefore eventually the constant sequence $\{r_F(M)\}$.

To complete the proof it is enough to show that $r_F(M) = \min\{r_F(M_P) \mid \operatorname{sr}(M_P) = 0\}$, which concludes (5). Moreover, since $\min\{r_F(M_P) \mid \operatorname{sr}(M_P) = 0\} > 0$ by Corollary 5.2, this also implies $a_e(M) = r_F(M) > 0$ for $e \gg 0$. Hence $\operatorname{sr}(M) = 0$, which concludes the proof of parts (1), (2), and (3).

Let $\{P_1, \dots, P_s\}$ be the set of maximal objects, with respect to containment, of the set of all centers of F-purity of R . We refer to them as the maximal centers of F-purity of R . We may assume that $\text{sr}(M_{P_i}) = 0$ for all $1 \leq i \leq r$, and $\text{sr}(M_{P_i}) > 0$ for $r+1 \leq i \leq s$. From what shown above, we know that for all $e \gg 0$ we have $F_*^e M \cong R^{\oplus r_F(M)} \oplus M_e$, where $F_*^{e'} M_e$ does not have a free summand for all $e' \geq 0$. We claim that $\text{frk}((M_e)_{P_i}) \geq 1$ for all $r+1 \leq i \leq s$. To see this, we assume by contradiction that, for some $r+1 \leq i \leq s$, we have $\text{frk}((M_e)_{P_i}) = 0$ for infinitely many $e \in \mathbb{Z}_{>0}$. Then the splitting rate of M_{P_i} would be 0, and this contradicts our arrangement of the maximal centers of F-purity of R .

Suppose $r_F(M) < \min\{r_F(M_P) \mid \text{sr}(M_P) = 0\}$. Then $\text{frk}((M_e)_{P_i}) > 0$ for each $1 \leq i \leq r$, and $e \gg 0$. Then for each $1 \leq i \leq s$ we can find $m_i \in M_e$ and $h_i \in \text{Hom}_R(M_e, R)$ such that $h_i(m_i) \notin P_i$. By prime avoidance we can find for each $1 \leq i \leq s$ an element $r_i \in (\bigcap_{j \neq i} P_j) \setminus P_i$. Let $m = \sum r_i m_i$ and $h = \sum r_i h_i$. Then $x := h(m) = \sum \sum r_i r_j h_i(m_j) \notin \bigcup_{i=1}^s P_i$. Therefore the element x avoids all maximal centers of F-purity of R , hence all centers of F-purity of R .

In particular, if $Q \in \text{Spec}(R)$, then there exists $e_Q \in \mathbb{Z}_{>0}$ such that $R_Q \xrightarrow{\cdot F_*^{e'} x} F_*^{e'} R_Q$ splits for all $e' \geq e_Q$. Therefore, the union of the sets $U_{e'} := \{Q \in \text{Spec}(R) \mid R_Q \xrightarrow{\cdot F_*^{e'} x} F_*^{e'} R_Q \text{ splits}\}$ is equal to $\text{Spec}(R)$. Moreover, they are open sets, and they form an ascending chain [HH89]. By quasi-compactness of $\text{Spec}(R)$, there exists $e' \in \mathbb{Z}_{>0}$ such that $U_{e'} = \text{Spec}(R)$. Therefore $R \xrightarrow{\cdot F_*^{e'} x} F_*^{e'} R$ splits, since splitting of is a local condition. Suppose $\varphi : F_*^{e'} R \rightarrow R$ satisfies $\varphi(F_*^{e'} x) = 1$. Then, the composition $F_*^{e'} M_e \xrightarrow{\cdot F_*^{e'} h} F_*^{e'} R \xrightarrow{\varphi} R$ maps $F_*^{e'} h \mapsto 1$, and this contradicts the property that $F_*^{e'} M_e$ does not have a free R -summand for all $e' \geq 0$. This completes the proof. \square

We end this section by showing that the global F-splitting ratio of a positively graded algebra is equal to the F-splitting ratio at the irrelevant maximal ideal.

Proposition 5.7. *Let (R_0, \mathfrak{m}_0, k) be an F-finite local ring and let R be a positively graded algebra of finite type over R_0 . Let $R_{>0}$ be the ideal of R generated by elements of positive degree and $\mathfrak{m} = \mathfrak{m}_0 + R_{>0}$. Suppose that M is a finitely generated graded R -module. We have the equality $a_e(M) = a_e(M_{\mathfrak{m}})$. In particular, we have $\text{sr}(M) = \text{sr}(M_{\mathfrak{m}})$, and $r_F(M) = r_F(M_{\mathfrak{m}})$.*

Proof. Since $a_e(M) \leq a_e(M_{\mathfrak{m}})$ always holds, it is sufficient to prove the other inequality. To this end, we observe that $F_*^e M$ is a \mathbb{Q} -graded module. Hence, we can find a graded isomorphism $F_*^e M \cong \bigoplus_{i=1}^{b_e} R[n_i] \oplus M_e$, where $n_i \in \mathbb{Q}$, and M_e is a \mathbb{Q} -graded module with no graded free summands. Here, $R[n_i]$ denotes the cyclic \mathbb{Q} -graded free module whose generator is in degree $-n_i$. We claim that $(M_e)_{\mathfrak{m}}$ has no free summands either. In fact, if it did, there would be a surjective $R_{\mathfrak{m}}$ -linear map $(M_e)_{\mathfrak{m}} \rightarrow R_{\mathfrak{m}}$. Such a map lifts to an R -linear map $\varphi : M_e \rightarrow R$ with $\varphi(M_e) \not\subseteq \mathfrak{m}$. Since $\text{Hom}_R(M_e, R)$ is a graded module, we can find a graded component ψ of φ that still satisfies $\psi(M_e) \not\subseteq \mathfrak{m}$. Such a map ψ gives rise to a graded free summand of M_e , contradicting our assumptions. This shows that $a_e(M) \geq b_e = a_e(M_{\mathfrak{m}})$, as claimed. \square

Corollary 5.8. *Let R and \mathfrak{m} be as in Proposition 5.7. We have $s(R) = s(R_{\mathfrak{m}})$.*

Proof. In our assumptions, the ideal defining the non-strongly F-regular locus is homogeneous [LS99, Lemma 4.2]. If R is not strongly F-regular, then $R_{\mathfrak{m}}$ is also not strongly F-regular;

thus, $s(R) = s(R_{\mathfrak{m}}) = 0$ in this case. Now assume R is strongly F-regular. Then $R_{\mathfrak{m}}$ is also strongly F-regular, and thus $\text{sr}(R_{\mathfrak{m}}) = \gamma(R_{\mathfrak{m}}) = \gamma(R)$. Using Proposition 5.7, we conclude that $\text{sr}(R) = \gamma(R)$, and hence $s(R) = r_F(R) = r_F(R_{\mathfrak{m}}) = s(R_{\mathfrak{m}})$. \square

6. POSITIVITY OF F-SIGNATURE OF CARTIER ALGEBRAS AND STRONG F-REGULARITY

This section is devoted to giving a positive answer to [DSPY19b, Question 4.24]. We recall the following condition from [DSPY19b]. For unexplained notation and terminology we refer to Subsection 2.4 of the same article.

Condition 6.1. We say that (R, \mathcal{D}) satisfies condition (\dagger) if at least one of the following conditions is satisfied:

- \mathcal{D} satisfies condition $(*)$, as in 3.2.
- $\mathcal{D} = \mathcal{C}^{\mathfrak{a}^t}$ for some ideal $\mathfrak{a} \subseteq R$ and $t > 0$.
- R is normal and $\mathcal{D} = \mathcal{C}^{(R, \Delta)}$ for some effective \mathbb{Q} -divisor Δ .

Using the same notation as in Section 5, we now recall the definition of global F-signature of a pair (R, \mathcal{D}) . Given an F-finite and F-pure ring R , and a Cartier algebra \mathcal{D} , the F-signature of (R, \mathcal{D}) is

$$s(R, \mathcal{D}) = \lim_{e \rightarrow \infty} \frac{a_e(R, \mathcal{D})}{p^{e\gamma(R)}}.$$

When \mathcal{D} is the full Cartier algebra, we simply write $s(R)$ for $s(R, \mathcal{D})$. In this case, if we also have $\gamma(R) = \text{sr}(R)$, the global F-signature $s(R)$ coincides with the global F-splitting ratio $r_F(R)$ defined in Section 5. The limit above was shown to exist in [DSPY19b, Theorem 4.19]. In the same article, a global version of a result of Blickle, Schwede and Tucker [BST12], relating the positivity of $s(R, \mathcal{D})$ to the strong F-regularity of the pair (R, \mathcal{D}) was established in this setup.

Theorem 6.2. [DSPY19b, Corollary 4.23] *Let R be an F-finite domain, and let \mathcal{D} be a Cartier algebra satisfying condition (\dagger) . Then $s(R, \mathcal{D}) > 0$ if and only if (R, \mathcal{D}) is strongly F-regular*

The way Theorem 6.2 was proved in [DSPY19b] was by exploiting the relation

$$s(R, \mathcal{D}) = \min\{s(R_P, \mathcal{D}_P) \mid P \in \text{Spec}(R)\}.$$

Since the strong F-regularity of (R, \mathcal{D}) is equivalent to such minimum being positive, this was sufficient. However, the proof of the equality between the global F-signature of (R, \mathcal{D}) and the minimum of the local invariants required some semi-continuity results, that are only known to hold under the additional assumption that (\dagger) holds [Pol18, PT18]. The goal of this section is to show that Theorem 6.2 is true without assuming (\dagger) . In particular, we will provide a direct way to show that the signature of a strongly F-regular pair (R, \mathcal{D}) is positive, without looking at the corresponding invariants in the localizations at prime ideals.

We start with two preparatory lemmas.

Lemma 6.3. [BST12, Lemma 3.13c] and [PT18, Lemma 4.2] *Let R be an F-finite normal domain and let $\varphi \in \text{Hom}_R(F_*^e R, R)$. There exists $0 \neq z \in R$ such that for all $n \in \mathbb{Z}_{>0}$, and all $\psi \in \text{Hom}_R(F_*^{ne} R, R)$, there exists $r \in R$ such that*

$$z\psi = \varphi^n(F_*^{ne} r -)$$

where $\varphi^n = \varphi \circ F_*^e \varphi \circ F_*^{2e} \varphi \circ \dots \circ F_*^{(n-1)e} \varphi$ and $\varphi^n(F_*^e r -)$ is composition of the maps

$$F_*^n R \xrightarrow{F_*^{ne} r} F_*^{ne} R \xrightarrow{\varphi^n} R.$$

Lemma 6.4. *Let R be a strongly F-regular F-finite domain. Then there exists $\varepsilon > 0$ such that for all $e \in \mathbb{Z}_{>0}$, $a_e(R) \geq \varepsilon \operatorname{rank}(F_*^e R)$.*

Proof. As R is strongly F-regular, $s(R) > 0$ by [DSPY19b, Theorem 4.16]. Hence, there exists $e' \in \mathbb{Z}_{>0}$ such that for all $e > e'$, $a_e(R)/\operatorname{rank}(F_*^e R) \geq s(R)/2$. Let

$$\varepsilon = \min \left\{ \frac{a_1(R)}{\operatorname{rank}(F_* R)}, \dots, \frac{a_{e'}(R)}{\operatorname{rank}(F_*^{e'} R)}, \frac{s(R)}{2} \right\}.$$

Then $a_e(R) \geq \varepsilon \operatorname{rank}(F_*^e R)$ for all $e \in \mathbb{Z}_{>0}$. \square

The following theorem extends [DSPY19b, Theorem 2.24], giving a positive answer to [DSPY19b, Question 4.24].

Theorem 6.5. *Let R be an F-finite domain and let \mathcal{D} be a Cartier algebra. Then (R, \mathcal{D}) is strongly F-regular if and only if $s(R, \mathcal{D}) > 0$.*

Proof. If (R, \mathcal{D}) is not strongly F-regular, then there exists $P \in \operatorname{Spec}(R)$ such that (R_P, \mathcal{D}_P) is not strongly F-regular. Since $a_e(R, \mathcal{D}) \leq a_e(R_P, \mathcal{D}_P)$, we get $s(R, \mathcal{D}) \leq s(R_P, \mathcal{D}_P) = 0$.

Conversely, suppose that (R, \mathcal{D}) is strongly F-regular. Then R is strongly F-regular and by Lemma 6.4 there exists $\varepsilon > 0$ such that $a_e(R) \geq \varepsilon \operatorname{rank}(F_*^e R)$ for all $e \in \mathbb{Z}_{>0}$. Let $e_0 \in \mathbb{Z}_{>0}$ be such that $\varepsilon \geq \frac{1}{p^{e_0}}$. If $\operatorname{rank}(F_*^e R) = 1$ for each $e \in \mathbb{Z}_{>0}$, then R is a perfect field and there is nothing to prove. We assume R is not a perfect field so that, for all $e \geq e_0$, p^{e_0} divides $\operatorname{rank}(F_*^e R)$. Let $\ell_e = \operatorname{rank}(F_*^e R)/p^{e_0}$, so that $a_e(R) \geq \ell_e$ for each $e \in \mathbb{Z}_{>0}$.

Let $e_1 > 0$ be such that $a_{e_1}(R, \mathcal{D}) > 0$, and let $\varphi \in \mathcal{D}_{e_1}$ be a non-zero map. Let z be as in Lemma 6.3. In particular, for each $n \in \mathbb{Z}_{>0}$ and for each $\psi \in \operatorname{Hom}_R(F_*^{ne_1} R, R)$, the map $z\psi$ belongs to \mathcal{D}_{ne_1} . Consider integers of the form $e = ne_1 \geq e_0$. As $a_e(R) \geq \ell_e$, we can write $F_*^e R \cong R^{\oplus \ell_e} \oplus M_e$ for some R -module M_e . Let $\lambda_1, \dots, \lambda_{\ell_e} \in F_*^e R$ form a basis for the free summand $R^{\oplus \ell_e}$ of $F_*^e R$. Denote by $\tilde{\lambda}_i : F_*^e R \rightarrow R$ the R -linear map defined by $\lambda_i \mapsto 1$, $\lambda_j \mapsto 0$ for all $j \neq i$, and $x \mapsto 0$ for all $x \in M_e$.

We chose $0 \neq z \in R$ such that $z\tilde{\lambda}_i \in \mathcal{D}_e$, and $z\tilde{\lambda}_i$ maps $\lambda_i \mapsto z$ and $\lambda_j \mapsto 0$ for all $j \neq i$. As (R, \mathcal{D}) is strongly F-regular, there exists $e_2 \in \mathbb{Z}_{>0}$ and $\gamma \in \mathcal{D}_{e_2}$ such that $\gamma(F_*^{e_2} z) = 1$. Then the R -linear maps $\gamma_i := \gamma \circ F_*^{e_2} z\tilde{\lambda}_i : F_*^{e_2+e} R \rightarrow R$ are elements of \mathcal{D}_{e_2+e} such that $F_*^{e_2} \lambda_i \mapsto 1$ and $F_*^{e_2} \lambda_j \mapsto 0$ for all $j \neq i$. Therefore, for each $e = ne_1 \geq e_0$, we have

$$a_{ne_1+e_2}(R, \mathcal{D}) \geq \ell_{ne_1} = \frac{\operatorname{rank}(F_*^{ne_1} R)}{p^{e_0}} = \frac{\operatorname{rank}(F_*^{ne_1+e_2} R)}{p^{e_0} \operatorname{rank}(F_*^{e_2} R)},$$

and thus

$$s(R, \mathcal{D}) = \lim_{e' \in \Gamma_{\mathcal{D}} \rightarrow \infty} \frac{a_{e'}(R, \mathcal{D})}{\operatorname{rank}(F_*^{e'} R)} = \lim_{n \rightarrow \infty} \frac{a_{ne_1+e_2}(R, \mathcal{D})}{\operatorname{rank}(F_*^{ne_1+e_2} R)} \geq \frac{1}{p^{e_0} \operatorname{rank}(F_*^{e_2} R)} > 0. \quad \square$$

Remark 6.6. As pointed out above, the proof of Theorem 6.2 contained in [DSPY19b] requires the extra assumption that (\dagger) holds, because it is based on the equality $s(R, \mathcal{D}) = \min\{s(R_P, \mathcal{D}_P) \mid P \in \operatorname{Spec}(R)\}$. Theorem 6.5 settles the positivity of $s(R, \mathcal{D})$ for strongly F-regular pairs (R, \mathcal{D}) , but it does not indicate any progress in the direction of showing that $s(R, \mathcal{D})$ is equal to the minimum of the local invariants. In particular, it does not show the existence of a prime $P \in \operatorname{Spec}(R)$ such that $s(R, \mathcal{D}) = s(R_P, \mathcal{D}_P)$.

REFERENCES

- [AE05] Ian M. Aberbach and Florian Enescu. The structure of F -pure rings. *Math. Z.*, 250(4):791–806, 2005. [1](#), [3](#), [6](#)
- [BST12] Manuel Blickle, Karl Schwede, and Kevin Tucker. F -signature of pairs and the asymptotic behavior of Frobenius splittings. *Adv. Math.*, 231(6):3232–3258, 2012. [1](#), [2](#), [3](#), [4](#), [6](#), [14](#)
- [DSPY18] Alessandro De Stefani, Thomas Polstra, and Yongwei Yao. Generalizing Serre’s splitting theorem and Bass’s cancellation theorem via free-basic elements. *Proc. Amer. Math. Soc.*, 146(4):1417–1430, 2018. [11](#)
- [DSPY19a] Alessandro De Stefani, Thomas Polstra, and Yongwei Yao. Global Frobenius Betti numbers and Frobenius Euler characteristics. *Preprint*, 2019. [2](#)
- [DSPY19b] Alessandro De Stefani, Thomas Polstra, and Yongwei Yao. Globalizing F -invariants. *Adv. Math.*, 350:359–395, 2019. [2](#), [14](#), [15](#)
- [EY11] Florian Enescu and Yongwei Yao. The lower semicontinuity of the Frobenius splitting numbers. *Math. Proc. Cambridge Philos. Soc.*, 150(1):35–46, 2011. [9](#)
- [HH89] Melvin Hochster and Craig Huneke. Tight closure and strong F -regularity. *Mém. Soc. Math. France (N.S.)*, (38):119–133, 1989. Colloque en l’honneur de Pierre Samuel (Orsay, 1987). [13](#)
- [HL02] Craig Huneke and Graham J. Leuschke. Two theorems about maximal Cohen-Macaulay modules. *Math. Ann.*, 324(2):391–404, 2002. [10](#)
- [HR76] Melvin Hochster and Joel L. Roberts. The purity of the Frobenius and local cohomology. *Advances in Math.*, 21(2):117–172, 1976. [3](#)
- [KM09] Shrawan Kumar and Vikram B. Mehta. Finiteness of the number of compatibly split subvarieties. *Int. Math. Res. Not. IMRN*, (19):3595–3597, 2009. [4](#), [8](#)
- [LS99] Gennady Lyubeznik and Karen E. Smith. Strong and weak F -regularity are equivalent for graded rings. *Amer. J. Math.*, 121(6):1279–1290, 1999. [13](#)
- [MS12] Lance Edward Miller and Karl Schwede. Semi-log canonical vs F -pure singularities. *J. Algebra*, 349:150–164, 2012. [11](#)
- [Pol18] Thomas Polstra. Uniform bounds in F -finite rings and lower semi-continuity of the F -signature. *Trans. Amer. Math. Soc.*, 370(5):3147–3169, 2018. [2](#), [5](#), [9](#), [14](#)
- [PT18] Thomas Polstra and Kevin Tucker. F -signature and Hilbert–Kunz multiplicity : a combined approach and comparison. *Algebra Number Theory*, 12(1):61–97, 2018. [14](#)
- [Sch09] Karl Schwede. F -adjunction. *Algebra Number Theory*, 3(8):907–950, 2009. [4](#), [8](#)
- [Sch10] Karl Schwede. Centers of F -purity. *Math. Z.*, 265(3):687–714, 2010. [4](#)
- [ST12] Karl Schwede and Kevin Tucker. A survey of test ideals. In *Progress in commutative algebra 2*, pages 39–99. Walter de Gruyter, Berlin, 2012. [12](#)
- [Sta82] J. T. Stafford. Generating modules efficiently over noncommutative rings. In *Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 34th Year (Paris, 1981)*, volume 924 of *Lecture Notes in Math.*, pages 72–88. Springer, Berlin-New York, 1982. [11](#)
- [Tuc12] Kevin Tucker. F -signature exists. *Invent. Math.*, 190(3):743–765, 2012. [1](#), [3](#)

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI GENOVA, VIA DODECANESO 35, 16146 GENOVA, ITALY

Email address: destefani@dima.unige.it

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UT 84102 USA

Email address: polstra@math.utah.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, GEORGIA STATE UNIVERSITY, ATLANTA, GEORGIA 30303, USA

Email address: yyao@gsu.edu