THE FROBENIUS COMPLEXITY OF A LOCAL RING OF PRIME CHARACTERISTIC

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Abstract. We introduce a new invariant for local rings of prime characteristic, called Frobenius complexity, that measures the abundance of Frobenius actions on the injective hull of the residue field of a local ring. We present an important case where the Frobenius complexity is finite, and prove that complete, normal rings of dimension two or less have Frobenius complexity less than or equal to zero. Moreover, we compute the Frobenius complexity for the determinantal ring obtained by modding out the 2×2 minors of a 2×3 matrix of indeterminates, showing that this number can be positive, irrational and depends upon the characteristic. We also settle a conjecture of Katzman, Schwede, Singh and Zhang on the infinite generation of the ring of Frobenius operators of a local normal complete $\mathbb{Q}$-Gorenstein ring.

1. Introduction

Throughout this paper R is a commutative Noetherian ring, often local, of positive characteristic p, where p is prime. Let $q = p^e$, where $e \in \mathbb{N} = \{0, 1, \ldots\}$. Consider the eth Frobenius homomorphism $F^e : R \to R$ defined $F(r) = r^q$, for all $r \in R$. For an R-module M, an eth Frobenius action (or Frobenius operator) on M is an additive map $\phi : M \to M$ such that $\phi(rm) = r^{p^e}\phi(m)$, for all $r \in R, m \in M$.

In recent years, there has been an interest in the study of the Frobenius actions on the local cohomology modules $H^i_{\mathfrak{m}}(R)$, $i \in \mathbb{N}$, and the injective hull of the residue field of a local ring $(R, \mathfrak{m}, k)$, denoted here by $E = E_R(k)$. Many applications to problems either coming from tight closure theory in commutative algebra or from positive characteristic algebraic geometry have been found this way. Lyubeznik and Smith have been naturally led to the study of rings of Frobenius operators in relation to the localization problem in tight closure theory, and asked whether the Frobenius ring of operators on E is finitely generated over R in [LS]. Katzman has shown that, in general, this is not true in [Ka]. Later, Àlvarez Montaner, Boix and Zarzuela showed that infinite generation is common even among nice classes of rings in [ABZ]. Other important aspects of the generation of $\mathscr{F}(E)$, including introducing the twisted construction, were studied by Katzman, Schwede, Singh and Zhang recently in [KSSZ].

The goal of our paper is to formulate a new invariant for rings of prime characteristic that describes the abundance of Frobenius operators naturally associated to the ring. This concept allows us to measure systematically the generation of the ring of Frobenius operators, finite or infinite. We will be mainly concerned with Frobenius operators on the injective hull

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of the residue field of the ring. An interesting byproduct of our work is that the phenomenon investigated here, the Frobenius complexity, appears to only be loosely connected to tight closure theory, although Frobenius operators were first studied in relation to it. This seems to suggest that the Frobenius complexity of a ring is a new feature of prime characteristic rings in addition to those coming from tight closure theory.

In this paper, we prove that the ring of Frobenius operators of a zero-dimensional ring is finitely generated, see Theorem 2.11. We answer positively a conjecture on $\mathbb{Q}$-Gorenstein rings posed by Katzman, Schwede, Singh and Zhang in [KSSZ], see Theorem 4.5. Under certain conditions on its anticanonical cover, we show that the Frobenius complexity of a local complete normal ring is finite, see Theorem 4.7. Then in Theorem 4.9, we show that the complexity of the ring is less than or equal to zero for rings of dimension at most two. Finally, in Theorem 5.6, we show that even nice rings, such as determinantal rings, which are Cohen-Macaulay and strongly F-regular, can have strictly positive complexity.

Let us review some of the main notation that will be used in this paper. For any $e \geqslant 0$, we let $R^{(e)}$ be the R-algebra defined as follows: as a ring $R^{(e)}$ equals R while the R-algebra structure is defined by $r \cdot s = r^q s$, for all $r \in R$, $s \in R^{(e)}$. Note that when R is reduced we have that $R^{(e)}$ is isomorphic to $R^{1/q}$ as R-algebras. Also, $R^{(e)}$ as an $R^{(e)}$-algebra is simply R as an R-algebra. Similarly, for an R-module M, we can define a new R-module structure on M by letting $r * m = r^{p^e} m$, for all $r \in R$, $m \in M$. We denote this R-module by $M^{(e)}$. For example, given an ideal I of R, we have $R^{(e)} \otimes_R R/I$ is (naturally isomorphic to) $R/I^{[q]}$, in which $I^{[q]}$ is the ideal of R generated by $\{x^q : x \in I\}$.

Consider now an eth Frobenius action on M, $\phi : M \to M$. This map can naturally be identified with an R-module homomorphism $\phi : M \to M^{(e)}$. It can be seen that such an action naturally defines an R-module homomorphism $f_\phi : R^{(e)} \otimes_R M \to M$, where $f_\phi(r \otimes m) = r\phi(m)$, for all $r \in R$, $m \in M$. Here, $R^{(e)}$ has the usual structure as an R-module given by $R^{(e)} = R$ on the left, while on the right we have the twisted Frobenius action.

Let $\mathscr{F}^e(M)$ be the collection of all eth Frobenius operators on M. We have a natural R-module isomorphism:

$$\mathscr{F}^e(M) = \operatorname{Hom}_R(M, M^{(e)}) \cong \operatorname{Hom}_R(R^{(e)} \otimes_R M, M),$$

defined by $P(\phi) = f_\phi$. The R-module structure on $\mathscr{F}^e(M)$ is given by $(r\phi)(x) = r\phi(x)$ for every $r \in R$, $\phi \in \mathscr{F}^e(M)$ and $x \in M$.

Note that P is additive and $P(r'\phi)(r \otimes m) = r((r'\phi)(m)) = r(r'\phi(m)) = rr'\phi(m) = r'(r\phi(m)) = r'P(\phi)(r \otimes m)$. And so $P(r'\phi) = r'P(\phi)$, for all $r \in R$, all $\phi \in \operatorname{Hom}_R(M, M^{(e)})$.

Definition 1.1. We define *the algebra of Frobenius operators* on M by

$$\mathscr{F}(M) = \oplus_{e \geqslant 0} \mathscr{F}^e(M).$$

The ring operation on $\mathscr{F}(M)$ is given by composition of functions (as multiplication). If $\phi \in \mathscr{F}^e(M)$, $\psi \in \mathscr{F}^{e'}(M)$ then $\phi\psi := \phi \circ \psi \in \mathscr{F}^{e+e'}(M)$. Note that $\phi\psi \neq \psi\phi$ in general.

The ring operation on $\mathscr{F}(M)$ defines a module structure $\mathscr{F}^e(M)$ over $\mathscr{F}^0(M) = \operatorname{End}_R(M)$. Since R maps canonically to $\mathscr{F}^0(M)$, this makes $\mathscr{F}^e(M)$ an R-module by restriction of scalars. Note that $(\phi \circ r)(m) = \phi(rm) = (r^q\phi)(m)$, for all $r \in R, m \in M$. Therefore,

$\phi r = r^q \phi$, for all $r \in R$, $\phi \in \mathscr{F}^e(M)$, with $q = p^e$. This indicates that, in general, $\mathrm{End}_R(M)$ is not in the center of $\mathscr{F}(M)$ and hence $\mathscr{F}(M)$ is not an R-algebra using the well established notion of an algebra.

2. The Complexity of a Noncommutative Graded Ring

Definition 2.1. Let $A = \oplus_{e \geqslant 0} A_e$ be a $\mathbb{N}$-graded ring, not necessarily commutative.

(1) Let $G_e(A) = G_e$ be the subring of A generated by the homogeneous elements of degree less than or equal to e. (So $G_0(A) = A_0$.) We agree that $G_{-1} = A_0$.
(2) We use $k_e = k_e(A)$ to denote the minimal number of homogeneous generators of G_e as a subring of A over A_0. (So $k_0 = 0$.) We agree that $k_{-1} = 0$. We say that A is *degree-wise finitely generated* if $k_e < \infty$ for all e.
(3) For a degree-wise finitely generated ring A, we say that a set X of homogeneous elements of A minimally generates A if for all e, $X_{\leqslant e} = \{a \in X : \deg(a) \leqslant e\}$ is a minimal set of generators for G_e with $k_e = |X_{\leqslant e}|$ for every $e \geqslant 0$. Also, let $X_e = \{a \in X : \deg(a) = e\}$.

Remark 2.2. Let $A = \oplus_{e \geqslant 0} A_e$ be a $\mathbb{N}$-graded ring, not necessarily commutative.

(1) Note that G_e is $\mathbb{N}$-graded and $G_e \subseteq G_{e+1}$, for all $e \geqslant 0$. Also, $(G_e)_i = A_i$ for all $0 \leqslant i \leqslant e$ and $(G_e)_{e+1} \subseteq A_{e+1}$. Moreover, both A_i and $(G_e)_i$ are naturally A_0-bimodules, for all i, e.
(2) Assume that X minimally generates A. Then $|X_e| = k_e - k_{e-1}$ for all $e \geqslant 1$.
(3) Every degree-wise finitely generated $\mathbb{N}$-graded ring admits a minimal generating set; see Proposition 2.3 next.

Proposition 2.3. *With the notations introduced above, let X be a set of homogeneous elements of A. Then*

(1) *The set X generates A as a ring over A_0 if and only if $X_{\leqslant e}$ generates G_e as a ring over A_0 for all $e \geqslant 0$ if and only if the image of X_e generates $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule for all $e \geqslant 0$.*
(2) *Assume that A is degree-wise finitely generated $\mathbb{N}$-graded ring and X generates A as a ring over A_0. The set X minimally generates A as a ring over A_0 if and only if $|X_e|$ is the minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule for all $e \geqslant 0$.*

Proof. Both statements follow from consideration of degree. Here is a proof with details.

(1) Assume that X generates A as a ring over A_0. For any $e \geqslant 0$ and any $a \in A_e$, by considering degree, we see that a can be written as an expression involving elements in $A_0 \cup X_{\leqslant e}$. Consequently

$$a \in A_0 X_e A_0 + (G_{e-1})_e,$$

in which $A_0 X_e A_0$ stands for the A_0-bimodule generated by X_e. Thus $A_e = A_0 X_e A_0 + (G_{e-1})_e$, which verifies that the image of X_e generates $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule, for all $e \geqslant 0$.

Conversely, assume that the image of X_e generates $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule for all $e \geqslant 0$. It follows that, for any $e \geqslant 0$ and any $a \in A_e$, we have

$$a \in A_0 X_e A_0 + (G_{e-1})_e,$$

which implies that a is generated by $X_{\leqslant e}$ over A_0. Therefore A, as a ring, is generated by X over A_0.

(2) Suppose that there exists $e \geqslant 0$ such that $|X_{\leqslant e}| > k_e$. We may further assume that this e is minimal with the property. If $e = 0$, then $|X_{\leqslant 0}| > k_0 = 0$ and hence $|X_{\leqslant 0}|$ is greater than the minimal number of generators of $\frac{A_0}{(G_{-1})_0} = 0$. So assume $e > 0$. Then $|X_{\leqslant e-1}| = k_{e-1} < \infty$, hence

$$|X_e| = |X_{\leqslant e}| - |X_{\leqslant e-1}| > k_e - k_{e-1}.$$

Choose a set Y of homogeneous elements such that Y (minimally) generates G_e with $|Y| = k_e$. It follows that $Y_{\leqslant e-1}$ generates G_{e-1}, so that $|Y_{\leqslant e-1}| \geqslant k_{e-1}$. (In fact $|Y_{\leqslant e-1}| = k_{e-1}$.) Thus

$$|X_e| > k_e - k_{e-1} \geqslant |Y| - |Y_{\leqslant e-1}| = |Y_{\leqslant e}| - |Y_{\leqslant e-1}| = |Y_e|.$$

Applying (1) to G_e and Y, we see that the image of Y_e generates $\frac{(G_e)_e}{(G_{e-1})_e} = \frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule. This shows that $|X_e|$ is not the minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule.

Conversely, suppose that, for some $e \geqslant 0$, $|X_e|$ is not (hence greater than) the minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule. The case $e = 0$ is trivial. So we assume $e > 0$. There exists $Y_e \subseteq A_e$ with $|Y_e| < |X_e|$ such that Y_e consists of homogeneous elements and the image of Y_e generates $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule. By (1) applied to G_e, we see that $X_{\leqslant e-1} \cup Y_e$ generates G_e. Therefore, either $|X_{\leqslant e}| = \infty > k_e$ or $|X_{\leqslant e}| > |X_{\leqslant e-1}| + |Y_e| \geqslant k_e$, implying that X does not minimally generate A as a ring over A_0. □

Corollary 2.4. *Let A be a degree-wise finitely generated $\mathbb{N}$-graded ring and X a set of homogeneous elements of A. Then*

1. *The minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as an A_0-bimodule is $k_e - k_{e-1}$ for all $e \geqslant 0$.*
2. *If X is generates A as a ring over A_0 then $|X_e| \geqslant k_e - k_{e-1}$ for all $e \geqslant 0$.*

Proof. Both (1) and (2) follow from Proposition 2.3. □

From the above, we see that the sequence $\{k_e\}_e$ or $\{k_e - k_{e-1}\}_e$ describes how far away A is from being finitely generated over A_0. This leads us to define the *complexity* of A as follows. Recall that, for sequences $\{a_e\}_e$ and $\{a'_e\}_e$, we write $a_e = O(a'_e)$ precisely when there exists $b \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $|a_e| \leqslant |ba'_e|$ for all $e \geqslant k$.

Definition 2.5. Let A be a degree-wise finitely generated ring. The sequence $\{k_e\}_e$ is called the *growth* sequence for A. The *complexity* sequence is given by $\{c_e(A) = k_e - k_{e-1}\}_{e \geqslant 0}$. The *complexity* of A is

$$\inf\{n \in \mathbb{R}_{>0} : c_e(A) = k_e - k_{e-1} = O(n^e)\}$$

and it is denoted by $\mathrm{cx}(A)$. If there is no $n > 0$ such that $c_e(A) = O(n^e)$, then we say that $\mathrm{cx}(A) = \infty$.

Remark 2.6. Let A be as above.

(1) Note that $\operatorname{cx}(A) > 0$ implies $\operatorname{cx}(A) \geqslant 1$. (Indeed, if $0 < n < 1$, then $n^e \to 0$ as $e \to \infty$. Thus $c_e(A) = k_e - k_{e-1} = O(n^e)$ with $0 < n < 1$ implies that $c_e(A)$ is eventually zero.)

(2) It is obvious that $\operatorname{cx}(A) = 0$ if and only if the sequence $\{c_e(A)\}_{e\geqslant 0}$ is eventually zero if and only if A is finitely generated as a ring over A_0.

(3) Similarly, $\operatorname{cx}(A) = 1$ if the sequence $\{c_e(A)\}_{e\geqslant 0}$ is bounded by above, but not eventually zero.

(4) When R is d-dimensional, complete and S_2, Lyubeznik and Smith have showed that $\mathscr{F}(H^d_{\mathfrak{m}}(R))$ is generated by one element over $\mathscr{F}^0(H^d_{\mathfrak{m}}(R)) = R$, namely the canonical Frobenius action F on $H^d_{\mathfrak{m}}(R)$, see Example 3.7 in [LS]. This shows that $\operatorname{cx}(H^d_{\mathfrak{m}}(R)) = 0$ for d-dimensional S_2 local rings.

Definition 2.7. Let A and B be $\mathbb{N}$-graded rings and $h\colon A \to B$ be a graded ring homomorphism. We say that h is *nearly onto* if $B = B_0[h(A)]$ (that is, B as a ring is generated by $h(A)$ over B_0).

Theorem 2.8. *Let A and B be $\mathbb{N}$-graded rings that are degree-wise finitely generated. If there exists a graded ring homomorphism $h\colon A \to B$ that is nearly onto, then $c_e(A) \geqslant c_e(B)$ for all $e \geqslant 0$.*

Proof. Choose a set X of homogeneous elements of A such that X minimally generates A as a ring over A_0. Since $h\colon A \to B$ is nearly onto, we see

$$B = B_0[h(A)] = B_0[h(X)].$$

This implies that $h(X)$ is a set of homogeneous elements of B and moreover $h(X)$ generates B as a ring over B_0. By Corollary 2.4, we see

$$c_e(A) = |X_e| \geqslant |h(X_e)| = |(h(X))_e| \geqslant c_e(B)$$

for all e. ∎

Definition 2.9. Let A be a $\mathbb{N}$-graded ring such that there exists a ring homomorphism $R \to A_0$, where R is a commutative ring. We say that A is a (left) *R-skew algebra* if $aR \subseteq Ra$ for all homogeneous elements $a \in A$. A right R-skew algebra can be defined analogously. In this paper, our R-skew algebras will be left R-skew algebras and therefore we will drop the adjective "left" when referring it to them.

Corollary 2.10. *Let A be a degree-wise finitely generated R-skew algebra such that $R = A_0$. Then $c_e(A)$ equals the minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as a left R-module for all e.*

Proof. This follows from Corollary 2.4 and the fact that $A_0XA_0 = RXR = RX$ for any set X of homogeneous elements of A. ∎

An important example for us is the ring of Frobenius operators on an R-module M. Note that there exists a canonical homomorphism $R \to \mathscr{F}^0(M) = \operatorname{End}_R(M)$. The main example is the case of a complete local ring $(R, \mathfrak{m}, k)$ and $M = E = E_R(k)$, the injective hull of the residue field of R as an R-module. In this case $\mathscr{F}(E)$ is an R-skew algebra and $R = \mathscr{F}^0(E)$.

Next, we prove that $\mathscr{F}(E)$ is finitely generated over R when $\dim(R) = 0$. In fact, we will prove a more general result concerning $\mathscr{F}(M)$.

Theorem 2.11. *If M is an R-module with finite length, then $\mathscr{F}(M)$ is finitely generated over $\mathscr{F}^0(M) = \mathrm{Hom}_R(M, M)$ (and also over R via the natural ring homomorphism $R \to \mathrm{Hom}_R(M, M)$).*

Proof. By replacing R with $R/\mathrm{Ann}(M)$, we may assume that R is Noetherian with $\dim(R) = 0$. Then, as every such ring is a direct product of finitely many 0-dimensional local rings, we may further assume that $R = (R, \mathfrak{m}, k)$ is a Artinian local ring without loss of generality.

Fix a set of minimal generators $\{b_1, \ldots, b_r\}$ for M as an R-module, so that the images of b_i form a basis for $M/\mathfrak{m}M$.

There exists $e_0 \in \mathbb{N}$ such that $\mathfrak{m}^{[p^{e_0}]} \subseteq \mathrm{Ann}(M)$. (In fact $\mathfrak{m}^{[p^{e_0}]} = 0$, given that M is actually faithful after the consideration in the last paragraph.) Thus the maximal ideal $\mathfrak{m}$ annihilates $M^{(e)}$ for $e \geqslant e_0$.

Let e be an arbitrary integer such that $e \geqslant e_0$. By the last paragraph, every map in $\mathscr{F}^e(M) = \mathrm{Hom}_R(M, M^{(e)})$ factors through $M/\mathfrak{m}M$. Thus, for any (arbitrarily) chosen elements $m_j \in M$ with $j = 1, \ldots, r$, there is a (unique) map in $\mathscr{F}^e(M) = \mathrm{Hom}_R(M, M^{(e)})$ such that $b_j \mapsto m_j$ for all $j = 1, \ldots, r$. In particular, for any given $r \times r$ matrix $A = (a_{ij} \in M_{r\times r}(R)$, there is a (unique) map in $\mathscr{F}^e(M) = \mathrm{Hom}_R(M, M^{(e)})$ such that $b_j \mapsto \sum_{i=1}^r a_{ij} b_i$ for all $j = 1, \ldots, r$. (Here the expression $\sum_{i=1}^r a_{ij} b_i$ only involves the scalar multiplication of M, although it represents an element in $M^{(e)}$.) We agree to use $A^{(e)}$ to denote the map determined by A. To summarize, for every $A \in M_{r\times r}(R)$ there is a corresponding map $A^{(e)} \in \mathscr{F}^e(M)$, although different matrices could determine the same map. In particular, we have the map $\mathbf{I}^{(e)} \in \mathscr{F}^e(M)$ arising from the identity matrix $\mathbf{I}^{(e)} \in M_{r\times r}(R)$. (Note that this depends very much on the choice of $\{b_1, \ldots, b_r\}$. But we have already fixed $\{b_1, \ldots, b_r\}$ earlier.)

Also, it is clear that every map $\phi \in \mathscr{F}^e(M) = \mathrm{Hom}_R(M, M^{(e)})$ arises this way, i.e., $\phi = A^{(e)}$ for some $A \in M_{r\times r}(R)$; in fact this statement does not rely on $e \geqslant e_0$.

Moreover, for $\phi = A^{(e)} \in \mathscr{F}^e(M)$ and $\psi = B^{(e')} \in \mathscr{F}^{e'}(M)$, it is routine to verify that $\phi\psi = (AB^{[p^e]})^{e+e'}$, in which $B^{[p^e]}$ stands for the matrix derived from B by raising all entries of B to the p^e-th power. In short, $(A^{(e)})(B^{(e')}) = (AB^{[p^e]})^{(e+e')}$.

Now we are ready to prove that $\mathscr{F}(M)$ is finitely generated over $\mathscr{F}^0(M) = \mathrm{Hom}_R(M, M)$. Indeed, we claim that $\mathscr{F}(M)$, as a ring, is generated by $\mathscr{F}^0(M), \mathscr{F}^1(M), \ldots, \mathscr{F}^{2e_0-1}(M)$. (This would suffice, as each $\mathscr{F}^i(M)$ is a finitely generated left R-module via the natural ring homomorphism $R \to \mathrm{Hom}_R(M, M)$. This would also prove that $\mathscr{F}(M)$ is finitely generated over R.) Let $\phi \in \mathscr{F}^e(M)$ with $e \geqslant 2e_0$, so that $\phi = A^{(e)}$ for some $A \in M_{r\times r}(R)$. Since $e - e_0 \geqslant e_0$, we see

$$\phi = A^{(e)} = (A^{(e_0)})(\mathbf{I}^{(e-e_0)}).$$

(Here $A^{(e_0)} \in \mathscr{F}^{e_0}(M)$ and $\mathbf{I}^{(e-e_0)} \in \mathscr{F}^{e-e_0}(M)$ are regarded as Frobenius actions on M, as we set up above.) It suffices to verify that $\mathbf{I}^{(e-e_0)}$ can be generated by $\mathscr{F}^{e_0}(M), \ldots, \mathscr{F}^{2e_0-1}(M)$. If $e - e_0 \leqslant 2e_0 - 1$, this is clear. If $e - e_0 \geqslant 2e_0$, we write $e - e_0 = e_0 k + c$ with $1 \leqslant k \in \mathbb{Z}$ and $e_0 \leqslant c \leqslant 2e_0 - 1$, which implies

$$\mathbf{I}^{(e-e_0)} = (\mathbf{I}^{(e_0 k)})(\mathbf{I}^{(c)}) = \left[(\mathbf{I}^{(e_0)})^k\right](\mathbf{I}^{(c)}).$$

This completes the proof. □

Corollary 2.12. *If R is a 0-dimensional local ring, then $\mathscr{F}(E)$ is finitely generated over $R = \mathscr{F}^0(E)$.*

Definition 2.13. Let $(R, \mathfrak{m}, k)$ be a local ring. We define the *Frobenius complexity* of the ring R by

$$\mathrm{cx}_F(R) = \log_p(\mathrm{cx}(\mathscr{F}(E))).$$

Also, denote $k_e(R) := k_e(\mathscr{F}(E))$, for all e, and call these numbers the *Frobenius growth sequence* of R. Then $c_e = c_e(R) := k_e(R) - k_{e-1}(R)$ defines the *Frobenius complexity sequence* of R. If the Frobenius growth sequence of the ring R is eventually constant, then the Frobenius complexity of R is said to be $-\infty$. If $\mathrm{cx}(\mathscr{F}(E)) = \infty$, the Frobenius complexity of R is said to be ∞.

The reader should note that we will not regard $(R, \mathfrak{m}, k)$ as a degree-wise finitely generated ring, so the definition of k_e, c_e for R will not conflict with Definition 2.5.

The reason we take $\log_p$ in the above Definition is that there is substantial evidence that, in important cases, there is a limit as $p \to \infty$ of the Frobenius complexity. See Theorem 5.6 and subsequent work in Section 3.1 in [EY] leading up to a definition of Frobenius complexity in characteristic zero.

Remark 2.14.

(1) The Frobenius operator algebra $\mathscr{F}(E)$ is finitely generated over R if and only if the Frobenius complexity of the ring R equals $-\infty$.
(2) If the Frobenius complexity sequence is bounded but not eventually zero then the Frobenius complexity of the ring is 0.
(3) The completion R of a Stanley-Reisner ring has zero Frobenius complexity, when $\mathscr{F}(E)$ is not finitely generated, by results of Àlvarez Montaner, Boix and Zarzuela, namely [ABZ, Proposition 3.4 and 3.1.2].
(4) When $(R, \mathfrak{m}, k)$ is d-dimensional and Gorenstein, we have that $E_R(k) = H^d_{\mathfrak{m}}(R)$, and Remark 2.6 (2) shows that $\mathrm{cx}_F(R) = -\infty$.
(5) If $(R, \mathfrak{m}, k)$ is normal, $\mathbb{Q}$-Gorenstein and the order of the canonical module is relatively prime to p, $\mathscr{F}(E)$ is finitely generated over $\mathscr{F}^0(E)$ by [KSSZ, Proposition 4.1]. Hence $\mathrm{cx}_F(R) = -\infty$ in this case. (See Theorem 4.5 for the converse.)

3. T-construction

Katzman, Schwede, Singh and Zhang have introduced an important $\mathbb{N}$-graded ring in their paper [KSSZ], which is an example of an R-skew algebra. We will study the complexity of this skew-algebra in this section, and apply these results to the complexity of the ring R in subsequent sections.

First let us review the definition of this ring. Let $\mathscr{R}$ be an $\mathbb{N}$-graded commutative ring of prime characteristic p with $\mathscr{R}_0 = R$.

Definition 3.1. Let $T_e = \mathscr{R}_{p^e-1}$ and $T(\mathscr{R}) = \oplus_e T_e = \oplus_{e \geqslant 0} \mathscr{R}_{p^e-1}$. Define a ring structure on $T(\mathscr{R})$ by

$$a * b = ab^{p^e},$$

for all $a \in T_e$, $b \in T_{e'}$.

This operation together with the natural addition inherited from $\mathscr{R}$ defines a noncommutative $\mathbb{N}$-graded ring. Note that $T_0 = R$ and if $a \in T_e, r \in R$, then $a*r = ar^{p^e} = r^{p^e}a = r^{p^e}*a$, for all $e \geqslant 0$, and hence $T(\mathscr{R})$ is a skew R-algebra.

We are now interested in computing the complexity of $T(\mathscr{R})$, when $\mathscr{R} = R[x_1, \dots, x_d]$ a polynomial ring in d variables over R. Note that $T_e = \mathscr{R}_{p^e-1}$ is an R-free module with basis given by monomials of total degree $p^e - 1$.

We will use the notations introduced in the previous section. We see that $G_{e-1} = G_{e-1}(T(\mathscr{R}))$ is an R-free module with basis consisting of monomials that can be expressed as products (under $*$, the multiplication of $T(\mathscr{R})$) of monomials of degree $p^i - 1$ where $i \leqslant e-1$. So the R-basis of $(G_{e-1})_e$ consists of these monomials of total degree $p^e - 1$.

In conclusion the R-module $\frac{T_e}{(G_{e-1})_e}$ is R-free with a basis given by monomials of degree $p^e - 1$ which cannot be written as products of monomials of degree $p^i - 1$, with $i \leqslant e-1$. We will refer to this basis as the *monomial basis* of $\frac{T_e}{(G_{e-1})_e}$.

We introduce the following notation $c_{d,e} := \operatorname{rank}_R(\frac{T_e}{(G_{e-1})_e})$. By Corollary 2.10, we see that $c_{d,e} = c_e(T(\mathscr{R}))$.

Let $m = x_1^{a_1} \cdots x_d^{a_d}$ be a monomial in T_e, that is, a monomial of degree $p^e - 1$. This monomial m belongs to $(G_{e-1})_e$ if and only if it can be decomposed as $m = m' * m''$, where $m' \in T_{e_1}$, $m'' \in T_{e_2}$ with $1 \leqslant e_i < e$ and $e_1 + e_2 = e$.

In other words, $m \in (G_{e-1})_e$ if and only if there is a decomposition

$$m = (x_1^{a'_1} \cdots x_d^{a'_d}) * (x_1^{a''_1} \cdots x_d^{a''_d}) = \prod_{i=1}^{d} x_i^{a'_i + p^{e_1} \cdot a''_i},$$

for some $1 \leqslant e_1 < e$, $1 \leqslant e_2 < e$, $e_1 + e_2 = e$, $\sum a'_i = p^{e_1} - 1$, $\sum a''_i = p^{e_2} - 1$.

At this stage it is helpful to introduce several notations: For an integer $a \in \mathbb{N}$, if $a = c_n p^n + \cdots + c_1 p + c_0$ with $0 \leqslant c_i \leqslant p-1$ for all $0 \leqslant i \leqslant n$, then we use $a = \overline{c_n \cdots c_0}$ to denote the base p expression of a. Also, we write $a|_e$ to denote the remainder of a when divided by p^e. Thus, if $a = \overline{c_n \cdots c_0}$ then $a|_e = \overline{c_{e-1} \cdots c_0}$ and we refer to this number as the eth truncation of a.

Therefore, for $m = x_1^{a_1} \cdots x_d^{a_d} \in T_e$, there is a decomposition

$$m = (x_1^{a'_1} \cdots x_d^{a'_d}) * (x_1^{a''_1} \cdots x_d^{a''_d}) = \prod_{i=1}^{d} x_i^{a'_i + p^{e_1} \cdot a''_i},$$

for some $1 \leqslant e_1 < e$, $1 \leqslant e_2 < e$, $e_1 + e_2 = e$, $\sum a'_i = p^{e_1} - 1$, $\sum a''_i = p^{e_2} - 1$ if and only if there exists an integer $1 \leqslant e_1 \leqslant e-1$ such that

$$a_1|_{e_1} + \cdots + a_d|_{e_1} = p^{e_1} - 1.$$

It can readily be seen that this is further equivalent to

$$a_1|_{e_1} + \cdots + a_{d-1}|_{e_1} \leqslant p^{e_1} - 1,$$

by dropping the part involving a_d. (For the very last equivalence, the forward implication is trivial. For the backward direction, assume $a_1|_{e_1} + \cdots + a_{d-1}|_{e_1} \leqslant p^{e_1} - 1$, which readily yields

$$(\dagger) \qquad a_1|_{e_1} + \cdots + a_d|_{e_1} \leqslant p^{e_1} - 1 + a_d|_{e_1} \leqslant p^{e_1} - 1 + p^{e_1} - 1.$$

Also note that $a_i|_{e_1} \equiv a_i \mod p^{e_1}$ for all $i = 1, \ldots, d$. Thus the blanket assumption $a_1 + \cdots + a_d = p^e - 1$ implies

$$(\ddagger) \qquad a_1|_{e_1} + \cdots + a_d|_{e_1} \equiv (a_1 + \cdots + a_d)|_{e_1} = (p^e - 1)|_{e_1} \equiv p^{e_1} - 1 \mod p^{e_1}.$$

With (†) and (‡), the only possible choice for $a_1|_{e_1} + \cdots + a_d|_{e_1}$ is $p^{e_1} - 1$.)

In conclusion, we have the following

Proposition 3.2. *A monomial $m = x_1^{a_1} \cdots x_d^{a_d}$ in T_e gives an element of the monomial basis of $\frac{T_e}{(G_{e-1})_e}$ if and only if, for all $1 \leqslant e_1 < e$,*

$$a_1|_{e_1} + \cdots + a_{d-1}|_{e_1} + a_d|_{e_1} \geqslant p^{e_1}$$

if and only if, for all $1 \leqslant e_1 < e$,

$$a_1|_{e_1} + \cdots + a_{d-1}|_{e_1} \geqslant p^{e_1}.$$

Recall that for $a = \overline{c_n \cdots c_0}$, its e_1th truncation is

$$a|_{e_1} = c_{e_1-1}p^{e_1-1} + \cdots + c_0 = \overline{c_{e_1-1} \cdots c_0}.$$

This leads to the following reformulation of the result obtained above:

Proposition 3.3. *A monomial $m = x_1^{a_1} \cdots x_d^{a_d}$ of T_e gives an element of the monomial basis of $\frac{T_e}{(G_{e-1})_e}$ if and only if, for all $1 \leqslant e_1 < e$, the sum of the e_1th truncation of a_i, $1 \leqslant i \leqslant d$, carries over (to the digit corresponding to p^{e_1}) in base p if and only if, for all $1 \leqslant e_1 < e$, the sum of the e_1th truncation of a_i, $1 \leqslant i \leqslant d-1$, carries over (to the digit corresponding to p^{e_1}) in base p if and only if, for all $1 \leqslant e_1 \leqslant e-1$, the sum of a_i, $1 \leqslant i \leqslant d-1$, carries over (to the digit corresponding to p^{e_1}) in base p.*

Proposition 3.4. *Let $p \geqslant 2$ be prime and $d = 3$. Then*

$$c_{3,e} = \sum_{0 \leqslant n_0, \ldots, n_{e-1} \leqslant p-1} n_0(n_1+1) \cdots (n_{e-2}+1)n_{e-1} = (1/2)^e p^e (p-1)^2 (p+1)^{e-2}.$$

Proof. We are going to use Proposition 3.3 to find the number of $x^i y^j z^k \in T_e = (T(R[x,y,z]))_e$ that form the monomial basis of $\frac{T_e}{(G_{e-1})_e}$. Fix each $0 \leqslant i \leqslant p^e - 1$, write $i = \overline{c_{e-1} \cdots c_0}$. As $d - 1 = 2$, we need to find the number of choices for $j = \overline{a_{e-1} \cdots a_0}$ such that, when calculating $i + j$ in base p, there is positive carry over to the digit corresponding to p^{e_1} for all $1 \leqslant e_1 \leqslant e-1$. Since we are adding two numbers, we see that the positive carry over must be 1 precisely. By default ($i + j \leqslant p^e - 1$), there is no carry over to the digit corresponding to p^e. Thus, equivalently, we need to count the number of choices for $j = \overline{a_{e-1} \cdots a_0}$ such that

$$\begin{cases} c_0 + a_0 \geqslant p, \\ c_r + a_r \geqslant p - 1 \text{ for } 1 \leqslant r \leqslant e-2, \\ c_{e-1} + a_{e-1} \leqslant p - 2. \end{cases}$$

This number (depending on $i = \overline{c_{e-1}\cdots c_0}$) can be calculated to be

$$c_0(c_1+1)\cdots(c_{e-2}+1)(p-1-c_{e-1})$$

by examining the number of choices for each a_i, $0 \leqslant i \leqslant e-1$.

Finally, running over all choices of $i = \overline{c_{e-1}\cdots c_0}$, we have

$$\begin{aligned}
c_{3,e} &= \sum_{0\leqslant c_0,\dots,c_{e-1}\leqslant p-1} c_0(c_1+1)\cdots(c_{e-2}+1)(p-1-c_{e-1}) \\
&= \sum_{0\leqslant n_0,\dots,n_{e-1}\leqslant p-1} n_0(n_1+1)\cdots(n_{e-2}+1)n_{e-1} \qquad \text{(by relabeling)} \\
&= \sum_{n_0=0}^{p-1} n_0 \cdot \sum_{n_1=1}^{p} n_1 \cdots \sum_{n_{e-2}=1}^{p} n_{e-2} \cdot \sum_{n_{e-1}=0}^{p-1} n_{e-1} \\
&= (p(p-1)/2)^2((p+1)p/2)^{e-2} \\
&= (1/2)^e p^e (p-1)^2 (p+1)^{e-2}.
\end{aligned}$$

□

Corollary 3.5. *Let $p \geqslant 2$ be prime and $\mathscr{R} = R[x_1,x_2,x_3]$. Then $\operatorname{cx}(T(\mathscr{R})) = \frac{p(p+1)}{2} = \binom{p+1}{2}$.*

Proof. Note that by definition

$$\operatorname{cx}(T(\mathscr{R}) = \inf\{n > 0 : c_{3,e} = O(n^e)\}.$$

A quick check verifies the claim. □

Proposition 3.6. *We have the following formulas concerning $c_{d,e} = c_e(T(R[x_1,\dots,x_d]))$:*

1. *When $e = 1$, $c_{d,1} = \binom{d+p-2}{p-1}$.*
2. *If $0 \leqslant d \leqslant 2$, then $c_{d,e} = 0$ for all $e \geqslant 2$.*
3. *If $d \geqslant 3$, then for $e \geqslant 1$*

$$c_{d,e} \geqslant \sum_{i=0}^{p^e-1} \xi_e(i)\binom{d-3+i}{i},$$

where $\xi_e(i) = (p-1-c_{e-1})(c_{e-2}+1)\cdots(c_2+1)(c_1+1)c_0$ if $i = \overline{c_{e-1}\cdots c_0}$ in base p.

Proof. (1) By the definition of $c_{d,e}$, we see that $c_{d,1}$ is the number of monomials of degree $p-1$ in d variables, which is $\binom{d+p-2}{p-1}$.

(2) This follows from Proposition 3.2 or Proposition 3.3.

(3) First, recall that in the proof of Proposition 3.4 where $d = 3$, for each $e \geqslant 1$ and for each i between 0 and $p^e - 1$, there are $\xi_e(i)$ many monomials of the form $x^i y^j z^k$ that constitute part of the monomial basis of $\frac{T_e}{(G_{e-1})_e}$ for $T(R[x,y,z])$.

Second, note that if a monomial $x^i y^j z^k \in T(R[x,y,z])_e$ is in the monomial basis for $\frac{T_e}{(G_{e-1})_e}$ for $T(R[x,y,z])$ then any monomial $x_1^{a_1}\cdots x_{d-2}^{a_{d-2}} x_{d-1}^j x_d^k \in T(R[x_1,\dots,x_{d-2},x_{d-1},x_d])_e$ with $a_1+\cdots+a_{d-2} = i$ is in the monomial basis of $\frac{T_e}{(G_{e-1})_e}$ for $T(R[x_1,\dots,x_{d-2},x_{d-1},x_d])$. (To see this, study the contrapositive.)

Finally, observe that there are precisely $\binom{d-3+i}{i}$ many strings $(a_1, \dots, a_{d-2}) \in \mathbb{N}^{d-2}$ such that $a_1 + \cdots + a_{d-2} = i$.

Combining the above, we conclude that there are at least $\sum_{i=0}^{p^e-1} \xi_e(i)\binom{d-3+i}{i}$ many monomials in $T(R[x_1, \dots, x_{d-2}, x_{d-1}, x_d])_e$ that form a portion of the monomial basis of $\frac{T_e}{(G_{e-1})_e}$ for $T(R[x_1, \dots, x_{d-2}, x_{d-1}, x_d])$. Hence the inequality. □

Next, we are going to give another way to compute $c_{d,e} = c_e(T(\mathscr{R}))$ for $\mathscr{R} = R[x_1, \dots, x_d]$.

Proposition 3.7. *For $\mathscr{R} = R[x_1, \dots, x_d]$, we have the following formula for $c_{d,e} = c_e(T(\mathscr{R}))$:*

$$c_{d,e} = \sum_{\substack{(d_{e-1}=0,\, d_{e-2}, \dots, d_0,\, d_{-1}=0) \in \mathbb{N}^{e+1} \\ d_n > 0 \text{ for } 0 \leqslant n < e-1}} \prod_{n=0}^{e-1} M_d(d_n p - d_{n-1} + p - 1), \ \forall e \geqslant 1,$$

in which $M_d(m)$ stands for the rank of $(R[x_1, \dots, x_d]/(x_1^p, \dots, x_d^p))_m$ as an R-module.

Proof. It suffices to count the number of monomials that produce the monomial basis of $\frac{T_e}{(G_{e-1})_e}$, in which T is short for $T(\mathscr{R}) = T(R[x_1, \dots, x_d])$. Let $1 \leqslant e \in \mathbb{N}$ and $\underline{x}^{\underline{a}} := x_1^{a_1} \cdots x_d^{a_d} \in T_e$, in which $\underline{a} := (a_1, \dots, a_d) \in \mathbb{N}^d$ with $|\underline{a}| := a_1 + \cdots + a_d = p^e - 1$. For each $i \in \{1, \dots, d\}$, write $a_i = \overline{a_{i,e-1} \cdots a_{i,0}}$ in base p expression. Then, for each $0 \leqslant n \leqslant e-1$, denote $\underline{a}_n := (a_{1,n}, \dots, a_{d,n})$, which can be referred to as the vector of digits corresponding to p^n. Also, for each $0 \leqslant n \leqslant e-1$, let $d_n(\underline{a})$ denote the (accumulated) carry-over to the digit corresponding to p^{n+1} when computing $\sum_{i=1}^d a_i$ in base p. In other words, $d_n(\underline{a})$ is the carry-over to the digit corresponding to p^{n+1} when computing $a_1|_{n+1} + \cdots + a_d|_{n+1}$. Denote $d(\underline{a}) := (d_{e-1}(\underline{a}), \dots, d_0(\underline{a}))$. Note that $d_{e-1}(\underline{a}) = 0$ since $|\underline{a}| = a_1 + \cdots + a_d = p^e - 1$.

Given $\underline{x}^{\underline{a}} \in T_e$ and $\delta = (d_{e-1}, d_{e-2}, \dots, d_0)$ with $d_{e-1} = 0$, it is straightforward to see that $d(\underline{a}) = \delta$ if and only if

$$|\underline{a}_n| = d_n p - d_{n-1} + p - 1 \quad \text{for all} \quad n \in \{0, \dots, e-1\},$$

in which we agree that $d_{-1} = 0$.

By Proposition 3.3, the image of $\underline{x}^{\underline{a}}$ is an element of the monomial basis of $\frac{T_e}{(G_{e-1})_e}$ if and only if

$$d_n(\underline{a}) > 0 \quad \text{for} \quad 0 \leqslant n < e-1,$$

given that $|\underline{a}| = p^e - 1$ and (hence) $d_{e-1}(\underline{a}) = 0$.

Therefore, we can formulate $c_{d,e} = c_e(T(R[x_1, \dots, x_d]))$, $e \geqslant 1$, as follows (with the agreement $d_{-1} = 0 = d_{e-1}$ and $|\underline{a}| = p^e - 1$):

$$\begin{aligned}
c_{d,e} &= \sum_{\substack{(d_{e-1}=0,\, d_{e-2},\, \dots,\, d_0,\, d_{-1}=0) \in \mathbb{N}^{e+1} \\ d_n > 0 \text{ for } 0 \leqslant n < e-1}} \left| \{ \underline{a} \in \mathbb{N}^d : d(\underline{a}) = (d_{e-1}, d_{e-2}, \dots, d_0) \} \right| \\
&= \sum_{\substack{(d_{e-1}=0,\, d_{e-2},\, \dots,\, d_0,\, d_{-1}=0) \in \mathbb{N}^{e+1} \\ d_n > 0 \text{ for } 0 \leqslant n < e-1}} \left| \{ \underline{a} \in \mathbb{N}^d : |\underline{a}_n| = d_n p - d_{n-1} + p - 1 \text{ for } 0 \leqslant n \leqslant e-1 \} \right| \\
&= \sum_{\substack{(d_{e-1}=0,\, d_{e-2},\, \dots,\, d_0,\, d_{-1}=0) \in \mathbb{N}^{e+1} \\ d_n > 0 \text{ for } 0 \leqslant n < e-1}} \prod_{n=0}^{e-1} \left| \{ \underline{a}_n \in [0,\, p-1]^d : |\underline{a}_n| = d_n p - d_{n-1} + p - 1 \} \right| \\
&= \sum_{\substack{(d_{e-1}=0,\, d_{e-2},\, \dots,\, d_0,\, d_{-1}=0) \in \mathbb{N}^{e+1} \\ d_n > 0 \text{ for } 0 \leqslant n < e-1}} \prod_{n=0}^{e-1} M_d(d_n p - d_{n-1} + p - 1),
\end{aligned}$$

in which $[0, p-1] := \{0, \dots, p-1\}$ while, for any set X, $|X|$ stands for the cardinality of X. (Note that, in each of the summations above, almost all summands are zero.) □

Next, we outline a method that allows us compute $c_{d,e} = c_e(T(R[x_1,\, \dots,\, x_d]))$ for any $d \geqslant 3$, in which R may have any prime characteristic p. (Note that, if $d \leqslant 2$, then $c_{d,e} = 0$ for all $e \geqslant 2$.)

Discussion 3.8. Fix any $3 \leqslant d \in \mathbb{N}$, any prime number p, and any ring R with characteristic p. Let $\mathscr{R} = R[x_1,\, \dots,\, x_d]$. We can determine $c_{d,e} = c_e(T(\mathscr{R}))$ explicitly as follows:

Since $M_d(m)$ stands for the rank of $(R[x_1,\, \dots,\, x_d]/(x_1^p,\, \dots,\, x_d^p))_m$ over R, we see $M_d(m) = 0$ whenever $m > d(p-1)$ or $m < 0$. In fact, all $M_d(m)$ can be read off from the following Poincaré series (actually a polynomial):

$$\sum_{m=0}^{\infty} M_d(m) t^m = \left(\frac{1 - t^p}{1 - t} \right)^d = \left(1 + \cdots + t^{p-1} \right)^d.$$

By Proposition 3.7, we have

$$c_{d,e} = c_e(T(\mathscr{R})) = \sum_{\substack{(d_{e-1}=0,\, d_{e-2},\, \dots,\, d_0,\, d_{-1}=0) \in \mathbb{N}^{e+1} \\ d_n > 0 \text{ for } 0 \leqslant n < e-1}} \prod_{n=0}^{e-1} M_d(p d_n - d_{n-1} + p - 1),$$

in which the nonzero summands come from $(d_{e-1} = 0,\, d_{e-2},\, \dots,\, d_0,\, d_{-1} = 0) \in \mathbb{N}^{e+1}$ such that $d_n > 0$ and $0 \leqslant p d_n - d_{n-1} + p - 1 \leqslant d(p-1)$ for all $0 \leqslant n < e-1$.

From $d_n > 0$ and $0 \leqslant p d_n - d_{n-1} + p - 1 \leqslant d(p-1)$, $0 \leqslant n < e-1$, we see $0 < p d_n \leqslant (d-1)(p-1) + d_{n-1}$, which subsequently (and inductively) implies that there is a uniform upper bound for all possible d_n. In fact, it is not hard to see that

$$1 \leqslant d_n \leqslant d - 2 \quad \text{for all} \quad 0 \leqslant n < e - 1.$$

For every $e \geqslant 0$, denote

$$X_e = \begin{pmatrix} X_{e,1} \\ \vdots \\ X_{e,d-2} \end{pmatrix},$$

in which

$$X_{e,i} = \sum_{\substack{(d_e=i,\, d_{e-1},\, \dots,\, d_0,\, d_{-1}=0)\in\mathbb{N}^{e+2} \\ d_n\in\{1,\dots,d-2\} \text{ for } 0\leqslant n\leqslant e-1}} \prod_{n=0}^{e} M_d(pd_n - d_{n-1} + p - 1)$$

for all $i \in \{1, \dots, d-2\}$.

With these notations, it is straightforward to see that, for all $1 \leqslant i \leqslant d-2$,

$$X_{e+1,i} = \sum_{j=1}^{d-2} M_d(pi - j + p - 1) X_{e,j}$$

In other words, X_{e+1} can be computed recursively:

$$X_{e+1} = U \cdot X_e, \quad \text{where} \quad U := \big(u_{ij}\big)_{(d-2)\times(d-2)} \quad \text{and} \quad u_{ij} = M_d(pi - j + p - 1).$$

Therefore,

$$X_e = U^e \cdot X_0 \quad \text{for all} \quad e \geqslant 0.$$

With d and p given, both X_0 and $U = (u_{ij})_{(d-2)\times(d-2)}$ can be determined explicitly. Accordingly, we can compute $X_e = U^e \cdot X_0$ explicitly for all $e \geqslant 0$.

Finally, for all $e \geqslant 2$, we can determine $c_{d,e} = c_e(T(\mathscr{R}))$ explicitly, as follows:

$$\begin{aligned} c_{d,e} = c_e(T(\mathscr{R})) &= \sum_{\substack{(d_{e-1}=0,\, d_{e-2},\, \dots,\, d_0,\, d_{-1}=0)\in\mathbb{N}^{e+1} \\ d_n\in\{1,\dots,d-2\} \text{ for } 0\leqslant n\leqslant e-2}} \prod_{n=0}^{e-1} M_d(pd_n - d_{n-1} + p - 1) \\ &= \sum_{i=1}^{d-2} M_d(p\cdot 0 - i + p - 1) X_{e-2,i} = \sum_{i=1}^{d-2} M_d(p - i - 1) X_{e-2,i}. \end{aligned}$$

Consequently, $\operatorname{cx}(T(\mathscr{R}))$ can be computed.

To illustrate above method, we provide the following example, with $p = 2$ and $d = 4$.

Example 3.9. Let $\mathscr{R} = R[x_1, x_2, x_3, x_4]$, with R having characteristic 2. Since $M_4(m)$ stands for the rank of $(R[x_1, x_2, x_3, x_4]/(x_1^2, x_2^2, x_3^2, x_4^2))_m$ as an R-module, we see $M_4(0) = 1$, $M_4(1) = 4$, $M_4(2) = 6$, $M_4(3) = 4$, $M_4(4) = 1$, and $M_4(m) = 0$ for all $m > 4$ or $m < 0$.

Now, by Proposition 3.7, we have

$$c_{4,e} = c_e(T(\mathscr{R})) = \sum_{\substack{(d_{e-1}=0,\, d_{e-2},\, \dots,\, d_0,\, d_{-1}=0)\in\mathbb{N}^{e+1} \\ d_n>0 \text{ for } 0\leqslant n<e-1}} \prod_{n=0}^{e-1} M_4(2d_n - d_{n-1} + 1),$$

in which the nonzero summands are the ones coming from $(d_{e-1}, d_{e-2}, \dots, d_0, d_{-1}) \in \mathbb{N}^{e+1}$ such that $d_{-1} = d_{e-1} = 0$, $d_n > 0$, and $0 \leqslant 2d_n - d_{n-1} + 1 \leqslant 4$ for all $0 \leqslant n < e-1$.

From $0 \leqslant 2d_n - d_{n-1} + 1 \leqslant 4$, $0 \leqslant n < e-1$, we see $2d_n \leqslant 3 + d_{n-1}$, which subsequently shows that

$$1 \leqslant d_n \leqslant 2 \qquad \text{for all} \quad 0 \leqslant n < e-1.$$

For every $e \geqslant 0$, let us denote

$$A_e = \sum_{\substack{(d_e=1,\, d_{e-1},\, \dots,\, d_0,\, d_{-1}=0) \in \mathbb{N}^{e+2} \\ d_n \in \{1,2\} \text{ for } 0 \leqslant n \leqslant e-1}} \prod_{n=0}^{e} M_4(2d_n - d_{n-1} + 1),$$

$$B_e = \sum_{\substack{(d_e=2,\, d_{e-1},\, \dots,\, d_0,\, d_{-1}=0) \in \mathbb{N}^{e+2} \\ d_n \in \{1,2\} \text{ for } 0 \leqslant n \leqslant e-1}} \prod_{n=0}^{e} M_4(2d_n - d_{n-1} + 1).$$

With these notations, we see that A_e and B_e can be computed recursively:

$$A_{e+1} = M_4(2)A_e + M_4(1)B_e = 6A_e + 4B_e,$$
$$B_{e+1} = M_4(4)A_e + M_4(3)B_e = A_e + 4B_e.$$

Let $X_e = \begin{pmatrix} A_e \\ B_e \end{pmatrix}$ for all $e \geqslant 0$. So we have

$$X_{e+1} = U \cdot X_e, \quad \text{where} \quad U = \begin{pmatrix} 6 & 4 \\ 1 & 4 \end{pmatrix}.$$

By computing the eigenvalues and eigenvectors of U, we see that

$$U = P \cdot D \cdot P^{-1},$$

where

$$D = \begin{pmatrix} 5+\sqrt{5} & 0 \\ 0 & 5-\sqrt{5} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 1 & 1 \end{pmatrix}.$$

Therefore,

$$X_e = U^e \cdot X_0 = P \cdot D^e \cdot P^{-1} \cdot X_0 = \frac{1}{2\sqrt{5}} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} X_0,$$

in which

$$a_{11} = (1+\sqrt{5})(5+\sqrt{5})^e - (1-\sqrt{5})(5-\sqrt{5})^e,$$
$$a_{12} = 4(5+\sqrt{5})^e - 4(5-\sqrt{5})^e,$$
$$a_{21} = (5+\sqrt{5})^e - (5-\sqrt{5})^e,$$
$$a_{22} = (\sqrt{5}+1)(5+\sqrt{5})^e + (1+\sqrt{5})(5-\sqrt{5})^e.$$

But $X_0 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$. Accordingly, we get

$$\begin{pmatrix} A_e \\ B_e \end{pmatrix} = X_e = \frac{2}{\sqrt{5}} \begin{pmatrix} (1+\sqrt{5})(5+\sqrt{5})^e - (1-\sqrt{5})(5-\sqrt{5})^e \\ (5+\sqrt{5})^e - (5-\sqrt{5})^e \end{pmatrix}, \ \forall e \geqslant 0.$$

In conclusion, we see that, for all $e \geqslant 2$,

$$\begin{aligned} c_{4,e} = c_e(T(\mathscr{R})) &= \sum_{\substack{(d_{e-1}=0,\, d_{e-2},\, \dots,\, d_0,\, d_{-1}=0)\in\mathbb{N}^{e+1} \\ d_n\in\{1,2\} \text{ for } 0\leqslant n\leqslant e-2}} \prod_{n=0}^{e-1} M_4(2d_n - d_{n-1}+1) \\ &= M_4(0)A_{e-2} + M_4(-1)B_{e-2} = A_{e-2} \\ &= \frac{2}{\sqrt{5}}\left((1+\sqrt{5})(5+\sqrt{5})^{e-2} - (1-\sqrt{5})(5-\sqrt{5})^{e-2}\right). \end{aligned}$$

This shows that $\operatorname{cx}(T(\mathscr{R})) = 5+\sqrt{5}$, concluding the example.

4. Anticanonical cover

Throughout this section, we let $(R, \mathfrak{m}, k)$ be a local normal complete ring. For a divisorial ideal I, i.e., an ideal of pure height one, we denote $I^{(n)}$ its nth symbolic power. Let ω be a canonical ideal of R.

The anticanonical cover of the ring R is defined as

$$\mathscr{R} = \mathscr{R}(\omega) = \oplus_{n\geqslant 0}\omega^{(-n)}.$$

The following theorem was recently proved by Katzman, Schwede, Singh and Zhang in [KSSZ] and makes the transition from the T-construction to ring of Frobenius operators on injective hull of the residue field of R.

Theorem 4.1 (Katzman, Schwede, Singh, Zhang). *Let $(R, \mathfrak{m}, k)$ as above, E the R-injective hull of k and ω its canonical ideal. Then there exists a graded isomorphism:*

$$\mathscr{F}(E) \cong T(\mathscr{R}(\omega)).$$

An easy consequence of the above theorem is the following result, stated here for the convenience of the reader.

Proposition 4.2. *$\mathscr{F}(E)$ is principally generated over $\mathscr{F}^0(E)$ if and only if* $\operatorname{ord}(\omega) \mid p-1$.

In [KSSZ], it is shown that if R is $\mathbb{Q}$-Gorenstein with $p \nmid \operatorname{ord}(\omega)$, then $\mathscr{F}(E)$ is finitely generated over R. It is conjectured in [KSSZ] that if $p \mid \operatorname{ord}(\omega)$ then $\mathscr{F}(E)$ is not finitely generated over $\mathscr{F}^0(E)$.

We are going to prove the conjecture of [KSSZ], starting with a lemma and a corollary.

Lemma 4.3. *Let $I_1, \dots, I_m$ and $J \cong R$ be fractional ideals of a local domain $(R, \mathfrak{m})$.*

1. *If $I_1 \cdots I_m \cong R$ then $I_i \cong R$ for all $i = 1, \dots, m$.*
2. *Assume $I_1 \cdots I_m \subseteq J$. If $I_i \ncong R$ for some $i = 1, \dots, m$ then $I_1 \cdots I_m \subseteq \mathfrak{m}J$*

Proof. (1) If $\prod_{i=1}^m I_i \cong R$ then all I_i are projective and hence free (of rank one) over R.

(2) If $I_i \ncong R$ for some $i = 1, \dots, m$, then $\prod_{i=1}^m I_i \subsetneq J$ and hence $\prod_{i=1}^m I_i \subseteq \mathfrak{m}J$. □

We state the following corollary in such a way that it can be readily quoted in the proof of Theorem 4.5.

Corollary 4.4. *Let $(R,\mathfrak{m},k)$ be a $\mathbb{Q}$-Gorenstein normal domain with* $\operatorname{ord}(\omega) = m$. *Then for all $r \in \mathbb{Z}$ and $0 < s \in \mathbb{Z}$ such that $m \nmid r$ and $m \mid rs$, we have $(\omega^{(r)})^s \subseteq \mathfrak{m}\omega^{(rs)}$.*

Proof. This follows from Lemma 4.3, since $(\omega^{(r)})^s \subseteq \omega^{(rs)} \cong R$ and $\omega^{(r)} \not\cong R$. □

Theorem 4.5. *Let $(R,\mathfrak{m},k)$ be an excellent local normal domain with prime characteristic p. If R is $\mathbb{Q}$-Gorenstein with* $\operatorname{ord}(\omega) = m$ *such that $p \mid m$, then $\mathscr{F}(E)$ is not finitely generated over $(\mathscr{F}(E))_0 = \mathscr{F}^0(E)$ and* $\operatorname{cx}_F(R) = 0$.

Proof. We may assume R is complete. Since $\mathscr{F}(E) \cong T(\mathscr{R}(\omega))$ by [KSSZ], it suffices to prove that $T(\mathscr{R}(\omega))$ is not finitely generated over $(T(\mathscr{R}(\omega)))_0 = R$, in which $\mathscr{R}(\omega) = \oplus_{n\geqslant 0}\omega^{(-n)}$ with the (twisted) multiplication denoted by $*$. For shorter notation, denote

$$T := T(\mathscr{R}(\omega)) \quad \text{and} \quad T_e := (T(\mathscr{R}(\omega)))_e = \omega^{(1-p^e)}.$$

Now it suffices to prove that $T_{\leqslant e_0}$ does not generate T for any $e_0 \in \mathbb{N}$. To this end, fix an arbitrary $e_0 \in \mathbb{N}$; and let $G := G_{e_0}(T)$ denote the subring of T generated by $T_{\leqslant e_0}$. For every $e > e_0$ satisfying $p^{e-e_0} > m = \operatorname{ord}(\omega)$, we have the following:

$$\begin{aligned}
G_e &\subseteq \sum_{i=0}^{e_0}(T_{e-i} * T_i) = \sum_{i=0}^{e_0}\left(T_{e-i}T_i^{[p^{e-i}]}\right)\\
&= \sum_{i=0}^{e_0}\left(\omega^{(1-p^{e-i})}\big(\omega^{(1-p^i)}\big)^{[p^{e-i}]}\right)\\
&\subseteq \sum_{i=0}^{e_0}\left(\omega^{(1-p^{e-i})}\big(\omega^{(1-p^i)}\big)^{p^{e-i}}\right)\\
&= \sum_{i=0}^{e_0}\left(\omega^{(1-p^{e-i})}\big(\omega^{(1-p^i)}\big)^{p^{e-i}-m}\big(\omega^{(1-p^i)}\big)^{m}\right)\\
&\subseteq \sum_{i=0}^{e_0}\left(\omega^{(1-p^{e-i})}\big(\omega^{(1-p^i)}\big)^{p^{e-i}-m}\mathfrak{m}\omega^{((1-p^i)m)}\right) \quad \text{(by Corollary 4.4)}\\
&= \mathfrak{m}\sum_{i=0}^{e_0}\left(\omega^{(1-p^{e-i})}\big(\omega^{(1-p^i)}\big)^{p^{e-i}-m}\omega^{((1-p^i)m)}\right) \subseteq \mathfrak{m}\sum_{i=0}^{e_0}\omega^{(1-p^e)} = \mathfrak{m}T_e \subsetneq T_e.
\end{aligned}$$

Therefore $T_{\leqslant e_0}$ does not generate T. Thus $\operatorname{cx}_F(R) > -\infty$; see Remark 2.14(1).

To complete the proof, note that the Frobenius complexity sequence of R satisfies $c_e \leqslant \mu(\omega^{(1-p^e)})$ for all $e \geqslant 0$, in which $\mu(\omega^{(1-p^e)})$ stands for the minimal number of generators of $\omega^{(1-p^e)}$ over R. As $\operatorname{ord}(\omega) < \infty$, the Frobenius complexity sequence of R is bounded. Hence $\operatorname{cx}_F(R) = 0$; see Remark 2.14(2). □

As a corollary, we have the following result which deals with the total Cartier algebra of R. For a definition, we refer the reader to Definition 6.1 in [KSSZ].

Theorem 4.6. *Let $(R,\mathfrak{m},k)$ be an F-finite complete local normal $\mathbb{Q}$-Gorenstein domain of prime characteristic p with $p \mid$* $\operatorname{ord}(\omega)$. *Then the total Cartier algebra is not finitely generated over R. In fact, its complexity is* 1.

Proof. This is because the total Cartier algebra is isomorphic to the opposite of the Frobenius algebra as graded rings; see in [ABZ, 2.2.1], for example. ☐

We also state the following results on the complexity of a complete local normal ring.

Theorem 4.7. *Let $(R, \mathfrak{m}, k)$ be a local normal complete ring and further assume that $\mathscr{R} = \mathscr{R}(\omega)$ is a finitely generated R-algebra. Then $\operatorname{cx}_F(R) < \infty$. In fact, if $\mathscr{R}$ is generated by d many homogeneous elements over R, then $\operatorname{cx}_F(R) \leqslant d-1$.*

Proof. Suppose that $\mathscr{R}$ is finitely generated by homogeneous elements $r_1, \dots, r_d \in \mathscr{R}$ over R, with $\deg(r_i) = d_i$. We can find a surjective graded homomorphism:

$$\mathscr{S} = R[x_1, \dots, x_d] \to \mathscr{R},$$

in which we let $\deg(x_i) = d_i$, for all $1 \leqslant i \leqslant d$.

This induces a graded surjective homomorphism of skew algebras: $T(\mathscr{S}) \to T(\mathscr{R})$, and by Theorem 2.8 we get $c_e(T(\mathscr{S})) \geqslant c_e(T(\mathscr{R})) = c_e(R)$. So we only need to show $\operatorname{cx}(T(\mathscr{S})) < \infty$.

But by construction $T(\mathscr{S}) = \oplus_e \mathscr{S}_{p^e-1}$. According to our definitions

$$c_e(T(\mathscr{S})) \leqslant \mu_R(\mathscr{S}_{p^e-1}) \leqslant \binom{p^e-2+d}{d-1} = \frac{1}{(d-1)!} \cdot [p^e(p^e+1)\cdots(p^e+(d-2))].$$

This gives that

$$\operatorname{cx}(T(\mathscr{R})) \leqslant \operatorname{cx}(T(\mathscr{S})) \leqslant p^{d-1}.$$

So $\operatorname{cx}_F(R) \leqslant d-1$. ☐

Remark 4.8. The proof above indicates that, if an $\mathbb{N}$-graded commutative ring $\mathscr{R}$ of prime characteristic p is a finitely generated over $\mathscr{R}_0$ then $\operatorname{cx}(T(\mathscr{R})) < \infty$. In particular, if $\mathscr{R}$ is finitely generated by d many homogeneous elements over $\mathscr{R}_0$ then $\operatorname{cx}(T(\mathscr{R})) \leqslant p^{d-1}$.

For rings of dimension at most two, we can be more precise due to a result of Sally.

Theorem 4.9 (Sally, Theorem 1.2, page 51 and Theorem 2.1, page 52 in [Sa])**.** *Let $(R, \mathfrak{m}, k)$ be a local ring. Then*

(1) *The number of generators of all ideals is bounded above if and only if $\dim(R) \leqslant 1$;*
(2) *There is a bound on the number of generators of all ideals I such that $\mathfrak{m}$ is not an associated prime of I if and only if $\dim(R) \leqslant 2$.*

Theorem 4.10. *Let $(R, \mathfrak{m}, k)$ be a local, complete normal ring of dimension at most 2. Then $\operatorname{cx}_F(R) \leqslant 0$.*

Proof. When $\dim(R) \leqslant 1$, we know that R is regular and hence $\operatorname{cx}_F(R) = -\infty$ (see Remark 2.6 (iii)). Nevertheless, the following argument works for $\dim(R) \leqslant 2$.

According to Theorem 4.1, we have that $\mathscr{F}(E) \cong T(\mathscr{R}(\omega))$, where $\mathscr{R}(\omega)$ is the anticanonical cover of R. The graded parts of the anticanonical cover are ideals of pure dimension 1. By Theorem 4.9, the number of generators of these ideals are bounded by above. But Corollary 2.10 tells us now that $k_e(R)$ and hence $c_e(R)$ are bounded, and hence $\operatorname{cx}_F(R) \leqslant 0$. ☐

We close this section with the following relevant question.

Question 4.11. Let $(R, \mathfrak{m}, k)$ be a local complete normal ring. Is $\operatorname{cx}_F(R) < \infty$?

5. Examples of determinantal rings

Recall that, for $\mathbb{N}$-graded commutative rings $A = \oplus_{i\in\mathbb{N}} A_i$ and $B = \oplus_{i\in\mathbb{N}} B_i$ such that $A_0 = R = B_0$, their Segre product is

$$A \sharp B = \oplus_{i\in\mathbb{N}}(A_i \otimes_R B_i),$$

which is a ring under the natural operations.

In this section, we study the Segre product of $K[x_1, \ldots, x_d]$ and $K[y_1, \ldots, y_{d-1}]$. This ring is naturally isomorphic to the determinantal ring $K[X]/I$ where X is a $d \times (d-1)$ matrix of indeterminates and I is the ideal of the 2×2 minors of X.

Theorem 5.1 ([Wa, (4.2.3), page 430])**.** *Let K be a field and $d \geqslant 3$. The anticanonical cover of the Segre product of $K[x_1, \ldots, x_d]$ and $K[y_1, \ldots, y_{d-1}]$ is isomorphic to*

$$\bigoplus_{i\in\mathbb{N}} \left(\bigoplus_{\alpha\in\mathbb{N}^d,\, \beta\in\mathbb{N}^{d-1},\, |\alpha|-|\beta|=i} K x^\alpha y^\beta \right),$$

in which the grading is governed by i. Here, for $\alpha = (a_1, \ldots, a_d)$ and $\beta = (b_1, \ldots, b_{d-1})$ we denote $x^\alpha = x_1^{a_1} \cdots x_d^{a_d}$ and $y^\beta = y_1^{b_1} \cdots y_{d-1}^{b_{d-1}}$.

Remark 5.2.

(1) The Segre product of $K[x_1, \ldots, x_d]$ and $K[y_1, \ldots, y_{d-1}]$ can be described as follows

$$\begin{aligned} K[x_1, \ldots, x_d] \sharp K[y_1, \ldots, y_{d-1}] &= \bigoplus_{\alpha\in\mathbb{N}^d,\, \beta\in\mathbb{N}^{d-1},\, |\alpha|=|\beta|} K x^\alpha y^\beta \\ &\subset K[x_1, \ldots, x_d, y_1, \ldots, y_{d-1}]. \end{aligned}$$

(2) Let S_d denote the completion of $K[x_1, \ldots, x_d] \sharp K[y_1, \ldots, y_{d-1}]$ with respect to the ideal generated by all homogeneous elements of positive degree. It is easy to see that

$$\begin{aligned} S_d &\cong \prod_{\alpha\in\mathbb{N}^d,\, \beta\in\mathbb{N}^{d-1},\, |\alpha|=|\beta|} K x^\alpha y^\beta \\ &= \left\{ \sum_{\alpha\in\mathbb{N}^d,\, \beta\in\mathbb{N}^{d-1},\, |\alpha|=|\beta|} a_{\alpha,\beta} x^\alpha y^\beta \;\middle|\; a_{\alpha,\beta} \in K \right\} \subset K[[x_1, \ldots, x_d, y_1, \ldots, y_{d-1}]]. \end{aligned}$$

(3) Let $\mathscr{R}_d$ be the anticanonical cover of S_d. It follows from Theorem 5.1 that (with K being a field and $d \geqslant 3$)

$$\mathscr{R}_d \cong \bigoplus_{i\in\mathbb{N}} \left(\prod_{\alpha\in\mathbb{N}^d,\, \beta\in\mathbb{N}^{d-1},\, |\alpha|-|\beta|=i} K x^\alpha y^\beta \right),$$

in which the grading is governed by i.

Lemma 5.3. *Let A and B be degree-wise finitely generated $\mathbb{N}$-graded commutative rings and $h\colon A \to B$ be a graded ring homomorphism.*

(1) *The homomorphism h is nearly onto if and only if B_i is generated by $h(A_i)$ as a B_0-module for all $i \in \mathbb{N}$ (that is, B is generated by $h(A)$ as a B_0-module).*
(2) *If A and B have prime characteristic p and h is nearly onto, then the induced graded homomorphism $T(h)\colon T(A) \to T(B)$ is nearly onto.*

Proof. (1) It is clear that if B is generated by $h(A)$ as a B_0-bimodule then h is nearly onto, which does not rely on A or B being commutative.

Now assume h is nearly onto. Since B is commutative, it is routine to see that B_i is generated by $h(A_i)$ as a B_0-module for every $i \in \mathbb{N}$.

(2) If h is nearly onto, then B_i is generated by $h(A_i)$ as a B_0-module for every $i \in \mathbb{N}$. In particular, B_{p^e-1} is generated by $h(A_{p^e-1})$ as a B_0-module for every $e \in \mathbb{N}$. Viewing this inside $T(B)$, we see that $T(B)_e$ is generated by $T(h)(T(A)_e)$ as a left $T(B)_0$-module for every $e \in \mathbb{N}$. Therefore $T(h)$ is nearly onto (by part (1) above). □

Corollary 5.4. *Let A and B be $\mathbb{N}$-graded commutative rings of prime characteristic p. If there exists a graded ring homomorphism $h\colon A \to B$ that is nearly onto, then $c_e(T(A)) \geqslant c_e(T(B))$ for all $e \geqslant 0$.*

Proof. This follows from Lemma 5.3 and Theorem 2.8. □

Proposition 5.5. *Let S_d and $\mathscr{R}_d$ be as in Remark 5.2 with K being a field and $d \geqslant 3$. Then there are nearly onto graded ring homomorphisms from $\mathscr{R}_d$ to $K[x_1, \ldots, x_d]$ and vice versa.*

Proof. In light of Remark 5.2, we simply assume

$$\mathscr{R}_d = \bigoplus_{i\in\mathbb{N}} \left(\prod_{\alpha\in\mathbb{N}^d,\, \beta\in\mathbb{N}^{d-1},\, |\alpha|-|\beta|=i} Kx^\alpha y^\beta \right).$$

Define $\phi\colon \mathscr{R}_d \to K[x_1, \ldots, x_d]$ and $\psi\colon K[x_1, \ldots, x_d] \to \mathscr{R}_d$ by

$$\begin{aligned} \phi(f(x_1, \ldots, x_d, y_1, \ldots, y_{d-1})) &= f(x_1, \ldots, x_d, 0, \ldots, 0) \in K[x_1, \ldots, x_d] \\ \text{and} \qquad \psi(g(x_1, \ldots, x_d)) &= g(x_1, \ldots, x_d) \in \mathscr{R}_d, \end{aligned}$$

for all $f(x_1, \ldots, x_d, y_1, \ldots, y_{d-1}) \in \mathscr{R}_d$ and all $g(x_1, \ldots, x_d) \in K[x_1, \ldots, x_d]$.

It is routine to verify that both ϕ and ψ are graded ring homomorphisms. As $\phi \circ \psi$ is the identity map, we see that ϕ is onto and hence nearly onto. Finally, note that for every $i \in \mathbb{N}$, $(\mathscr{R}_d)_i$ is generated by $\psi((K[x_1, \ldots, x_d])_i)$ as a module over $(\mathscr{R}_d)_0 = S_d$. So ψ is nearly onto. This completes the proof. □

Theorem 5.6. *Let K, S_d and $\mathscr{R}_d$ be as in Remark 5.2 with $d \geqslant 3$. Assume that K has prime characteristic p. Then*

(1) *$T(\mathscr{R}_d)$ and $T(K[x_1, \ldots, x_d])$ have the same complexity sequence.*
(2) *$\operatorname{cx}_F(S_d) = \log_p \operatorname{cx}(T(K[x_1, \ldots, x_d]))$.*
(3) *$\mathscr{F}(E_d)$ is not finitely generated over $\mathscr{F}_0(E_d) \cong S_d$, where E_d stands for the injective hull of the residue field of S_d (over S_d).*
(4) *The Frobenius complexity of S_3 is $\operatorname{cx}_F(S_3) = 1 + \log_p(p+1) - \log_p 2$. Moreover, $\lim_{p\to\infty} \operatorname{cx}_F(S_3) = 2$.*

(5) *If* $p = 2$, *then* $\mathrm{cx}_F(S_4) = \log_2(5 + \sqrt{5})$.

Proof. Statements (1) and (2) follow from Corollary 5.4 and Proposition 5.5 in light of Theorem 4.1. Statement (3) follows from Proposition 3.6 or [KSSZ]. Finally, (4) follows from Corollary 3.5 while (5) follows from Example 3.9. □

Remark 5.7. (1) The statement in Theorem 5.6 (3) for $d = 3$ has been proved first in [KSSZ, Section 5]. It should be contrasted with (4) in the above Theorem 5.6.

(2) This computation shows that rings that are nice from the point of view of tight closure theory, such as completions of determinantal rings, can have positive Frobenius complexity. This means that, as $e \to \infty$, there are more and more eth Frobenius actions on $E_R(k)$ that are fundamentally new (i.e., do not come from Frobenius actions from lower degree). This phenomenon was illustrated in [KSSZ] as well, but the Frobenius complexity provides a way to quantify this. Interestingly, the Frobenius complexity can be irrational and depends upon the characteristic p. Also note that, although Gorenstein rings have Frobenius complexity $-\infty$, they can behave differently from the point of view of tight closure theory. In conclusion, the Frobenius complexity does not appear to be a measure of singularities in the same sense as other invariants coming from tight closure theory are.

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