FROBENIUS TEST EXPONENTS FOR PARAMETER IDEALS IN GENERALIZED COHEN–MACAULAY LOCAL RINGS

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Abstract. This paper studies Frobenius powers of parameter ideals in a commutative Noetherian local ring R of prime characteristic p. For a given ideal $\mathfrak{a}$ of R, there is a power Q of p, depending on $\mathfrak{a}$, such that the Q-th Frobenius power of the Frobenius closure of $\mathfrak{a}$ is equal to the Q-th Frobenius power of $\mathfrak{a}$. The paper addresses the question as to whether there exists a *uniform* Q_0 which 'works' in this context for all parameter ideals of R simultaneously.

In a recent paper, Katzman and Sharp proved that there does exists such a uniform Q_0 when R is Cohen–Macaulay. The purpose of this paper is to show that such a uniform Q_0 exists when R is a generalized Cohen–Macaulay local ring. A variety of concepts and techniques from commutative algebra are used, including unconditioned strong d-sequences, cohomological annihilators, modules of generalized fractions, and the Hartshorne–Speiser–Lyubeznik Theorem employed by Katzman and Sharp in the Cohen–Macaulay case.

0. Introduction

This paper studies a certain type of uniform behaviour of parameter ideals in a commutative Noetherian ring R of prime characteristic p.

One motivation for our work comes from the theory of test exponents for tight closure introduced by M. Hochster and C. Huneke in [8, Definition 2.2]. For an ideal $\mathfrak{a}$ of R and a non-negative integer n, the *p^n-th Frobenius power* $\mathfrak{a}^{[p^n]}$ of $\mathfrak{a}$ is the ideal of R generated by all p^n-th powers of elements of $\mathfrak{a}$. Suppose, temporarily, that R is reduced. Recall that a *test element* for R is an element c of R outside all the minimal prime ideals of R such that, for each ideal $\mathfrak{a}$ of R, and for $r \in R$, it is the case that $r \in \mathfrak{a}^*$, the tight closure of $\mathfrak{a}$, if and only if $cr^{p^n} \in \mathfrak{a}^{[p^n]}$ for all $n \geq 0$. It is a result of Hochster and Huneke [7, Theorem (6.1)(b)] that such a test element exists if R is a (reduced) algebra of finite type over an excellent local ring of characteristic p.

Let c be a test element for R, and let $\mathfrak{a}$ be an ideal of R. A *test exponent* for c, $\mathfrak{a}$ is a power $q = p^{e_0}$ (where e_0 is a non-negative integer) such that if, for an $r \in R$, we have $cr^{p^e} \in \mathfrak{a}^{[p^e]}$ for *one single* $e \geq e_0$, then $r \in \mathfrak{a}^*$ (so that $cr^{p^n} \in \mathfrak{a}^{[p^n]}$ for all $n \geq 0$). In [8], it is shown that this concept has strong connections with the major open problem about whether tight closure commutes with localization; indeed, to

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quote Hochster and Huneke, 'roughly speaking, test exponents exist if and only if tight closure commutes with localization'.

In a recent paper [19], R. Y. Sharp has shown that, for a test element c in a reduced equidimensional excellent local ring R (of characteristic p), there exists a non-negative integer e_0 such that p^{e_0} is a test exponent for c, $\mathfrak{a}$ for *every* parameter ideal $\mathfrak{a}$ of R. (In such an R, a parameter ideal is simply an ideal that can be generated by part of a system of parameters.) We can think of p^{e_0} as a *uniform parameter test exponent* for R.

It is natural to ask whether there is an analogous result for Frobenius closures. Return to the general situation where we assume only that R is a commutative Noetherian ring of prime characteristic p. The *Frobenius closure* $\mathfrak{a}^F$ of an ideal $\mathfrak{a}$ of R is defined by

$$\mathfrak{a}^F := \left\{ r \in R : \text{there exists } 0 \leq n \in \mathbb{Z} \text{ such that } r^{p^n} \in \mathfrak{a}^{[p^n]} \right\}.$$

This is an ideal of R, and so is finitely generated; therefore there exists a power Q_0 of p such that $(\mathfrak{a}^F)^{[Q_0]} = \mathfrak{a}^{[Q_0]}$, and we define $Q(\mathfrak{a})$ to be the smallest power of p with this property. Note that, for $r \in R$, it is the case that $r \in \mathfrak{a}^F$ if and only if $r^{Q(\mathfrak{a})} \in \mathfrak{a}^{[Q(\mathfrak{a})]}$.

In [10, §0], M. Katzman and Sharp raised the following question: is the set $\{Q(\mathfrak{b}) : \mathfrak{b} \text{ is a proper ideal of } R\}$ of powers of p bounded? In the case where R is Artinian, that is, has dimension 0, it is easy to see that this question has an affirmative answer, because in that case $R/\sqrt{0}$ is a direct product of fields, and one can deduce easily from that that, if Q_1 is a power of p such that $\left(\sqrt{0}\right)^{[Q_1]} = 0$, then $Q(\mathfrak{b}) \leq Q_1$ for every ideal $\mathfrak{b}$ of R. For this reason, we shall assume that $\dim R > 0$ for the remainder of the paper.

H. Brenner [1] has recently shown that the answer to the question of Katzman and Sharp, as stated above, is negative. Nevertheless, it might not be too unreasonable to hope that, in the case where R is local, the set

$$\{Q(\mathfrak{b}) : \mathfrak{b} \text{ is a parameter ideal of } R\}$$

is bounded. In [10, Theorem 2.5], Katzman and Sharp showed that this is the case when R is a Cohen–Macaulay local ring: they showed that, then, there exists an invariant $\eta(R)$ of R such that $(\mathfrak{b}^F)^{[p^{\eta(R)}]} = \mathfrak{b}^{[p^{\eta(R)}]}$ for all parameter ideals $\mathfrak{b}$ of R. The purpose of this paper is to prove the corresponding result when R is a generalized Cohen–Macaulay local ring, that is, when all the local cohomology modules $H^i_{\mathfrak{m}}(R)$ $(i = 0, \ldots, t-1)$ (where $\mathfrak{m}$ denotes the maximal ideal of R and $t := \dim R$) have finite length. Specifically, in Theorem 5.2 we prove the following.

Theorem. *Let R be a generalized Cohen–Macaulay local ring of prime characteristic p. Then there exists a power Q of p such that $((\mathfrak{b})^F)^{[Q]} = \mathfrak{b}^{[Q]}$ for every ideal $\mathfrak{b}$ of R that can be generated by part of a system of parameters of R.*

In the Cohen–Macaulay case, the invariant $\eta(R)$ was defined by means of the Hartshorne–Speiser–Lyubeznik Theorem (see [10, Theorem 1.4]) about a certain type of uniform behaviour of a left module over the Frobenius skew polynomial ring (associated to R) that is Artinian as an R-module: Katzman and Sharp applied this Hartshorne–Speiser–Lyubeznik Theorem to the top local cohomology module of a Cohen–Macaulay local ring.

In this paper, we make similar use of the Hartshorne–Speiser–Lyubeznik Theorem, although we apply it to all the local cohomology modules of the generalized Cohen–Macaulay local ring R. We also use a variety of other concepts and techniques from commutative algebra, including unconditioned strong d-sequences and work of S. Goto and K. Yamagishi [4] about them, filter-regular sequences, cohomological annihilators, and modules of generalized fractions.

Other motivation for this work is provided in [10, §0]. There is no doubt in our minds that uniform behaviour of Frobenius closures of the type established in this paper is both desirable for its own sake and also relevant to the vigorous and ongoing development of tight closure theory.

1. Notation and terminology

Throughout this paper, R will denote a Noetherian commutative ring with $\dim R = t > 0$, and $\mathfrak{a}$ will denote an ideal of R. We shall use $\operatorname{Var}(\mathfrak{a})$ to denote the *variety of* $\mathfrak{a}$; thus $\operatorname{Var}(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$. We shall use $\min(R)$ to denote the set of minimal prime ideals of R and R° to denote $R \setminus \bigcup_{\mathfrak{p}\in\min(R)} \mathfrak{p}$. The annihilator of an R-module M will be denoted by $\operatorname{Ann}_R(M)$. We shall sometimes use the notation $(R, \mathfrak{m})$ to indicate that R is local with maximal ideal $\mathfrak{m}$; then, $(\widehat{R}, \widehat{\mathfrak{m}})$ will denote the $\mathfrak{m}$-adic completion of R. Also in the local case, we say that $\underline{x} = x_1, \ldots, x_l$ is a *system of parameters of* $(R, \mathfrak{m})$ if $\sum_{i=1}^{l} x_i R$ is $\mathfrak{m}$-primary and $l = t$; we say $\underline{x}$ is a *subsystem of parameters of* $(R, \mathfrak{m})$ if it is a subsequence of a system of parameters of R.

Notation 1.1. Throughout the paper, $\underline{x} = x_1, x_2, \ldots, x_l$ will denote a sequence of l elements of R.

(i) We use $\mathbb{N}$ to denote the set of all non-negative integers, and $\mathbb{N}_+$ to denote the set of all positive integers.

(ii) The main results concern the case where R has prime characteristic p, but this hypothesis will only be in force when explicitly stated; then, q, q', Q, $\widetilde{Q}$ and Q_i ($i \in \mathbb{N}$) will always denote powers of p with non-negative integer exponents.

(iii) For integers $i \le j$, we denote the subset $\{i, \ldots, j\}$ of $\mathbb{Z}$ by $[i, j]$, and we agree that $[i, j] = \emptyset$ if $i > j$.

(iv) We adopt the normal convention that $a^0 = 1$ for all $a \in R$.

(v) For each $\emptyset \ne \Lambda \subseteq [1, l]$, we set $x_\Lambda := \prod_{i\in\Lambda} x_i \in R$. In case $\Lambda = \emptyset$, we agree that $x_\emptyset^n = 1$ and $\sum_{i\in\emptyset} x_i^n R = (0)$ for all $n \in \mathbb{N}$.

(vi) For any $\Lambda \subseteq [1, l]$ and $n_1, \ldots, n_l \in \mathbb{N}$, the sequence

$$\left(\left(\left(\textstyle\sum_{i\in\Lambda} x_i^{n_i+j} R\right) : x_\Lambda^j\right)\right)_{j=0,1,2,\ldots}$$

forms an ascending chain of ideals, and we denote its ultimate constant value by $\left(\sum_{i\in\Lambda} x_i^{n_i} R\right)^{\lim}$. Thus

$$\left(\textstyle\sum_{i\in\Lambda} x_i^{n_i} R\right)^{\lim} = \bigcup_{j\in\mathbb{N}} \left(\left(\textstyle\sum_{i\in\Lambda} x_i^{n_i+j} R\right) : x_\Lambda^j\right).$$

In particular $\left(\sum_{i\in\emptyset} x_i R\right)^{\lim} = (0)$.

(vii) For any $\Lambda \subseteq [1, l]$ and $n_1, \ldots, n_l \in \mathbb{N}$, we set

$$\left(\textstyle\sum_{i\in\Lambda} x_i^{n_i} R\right)^{(\underline{x})\text{-unm}} := \left(\left(\textstyle\sum_{i\in\Lambda} x_i^{n_i} R\right) : \textstyle\sum_{i\in[1,l]\setminus\Lambda} x_i R\right)$$

and refer to this as the *unmixed part of* $\sum_{i\in\Lambda} x_i^{n_i}R$ *relative to the sequence* $\underline{x} = x_1, x_2, \ldots, x_l$.

Next, we recall some definitions of various concepts in commutative algebra.

Definition 1.2. Recall that $\mathfrak{a}$ denotes an ideal of R.

(i) We say that $\underline{x}$ is an $\mathfrak{a}$-*filter regular sequence* if there exists an integer $n \in \mathbb{N}$ such that $\mathfrak{a}^n \subseteq \operatorname{Ann}_R\big(\big(\big(\sum_{i=1}^{j-1} x_iR\big) : x_j\big)/\sum_{i=1}^{j-1} x_iR\big)$ for all $j = 1, 2, \ldots, l$. It is easy to see that $\underline{x}$ is an $\mathfrak{a}$-filter regular sequence if and only if, for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Var}(\mathfrak{a})$, the natural images of $x_1, \ldots, x_l$ form a possibly improper regular sequence in $R_{\mathfrak{p}}$.

(ii) We say that $\underline{x}$ is a *d-sequence* if

$$\big(\big(\textstyle\sum_{i=1}^{j} x_iR\big) : x_{j+1}x_k\big) = \big(\big(\textstyle\sum_{i=1}^{j} x_iR\big) : x_k\big)$$

for all j, k such that $0 \le j < k \le l$.

(iii) If $x_1^{n_1}, \ldots, x_l^{n_l}$ form a d-sequence in any order and for any positive integers $n_1, n_2, \ldots, n_l$, then we say that $\underline{x}$ is an *unconditioned strong d-sequence*, or a *d^+-sequence*.

(iv) Generalized Cohen–Macaulay local rings were studied by P. Schenzel in [16] (where they were called 'quasi-Cohen–Macaulay local rings' ([16, Definition 2])) and by Schenzel, N. V. Trung and N. T. Cuòng in [18]. When $(R, \mathfrak{m})$ is local, we say that R is a *generalized Cohen–Macaulay local ring* if $\bigcap_{i=0}^{t-1} \operatorname{Ann}_R(H^i_{\mathfrak{m}}(R))$ (recall that t denotes $\dim R$) contains an $\mathfrak{m}$-primary ideal; since all the local cohomology modules of R with respect to $\mathfrak{m}$ are Artinian, this is the case if and only if $H^i_{\mathfrak{m}}(R)$ has finite length for all $i = 0, \ldots, \dim R - 1$. Note that R is a generalized Cohen–Macaulay local ring if and only if its completion $\widehat{R}$ is.

(v) Again when $(R, \mathfrak{m})$ is local, we say that R is *is Cohen–Macaulay on the punctured spectrum* if $R_{\mathfrak{p}}$ is Cohen–Macaulay for every $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$.

Notation 1.3. Suppose that R has prime characteristic p. In these circumstances, we shall always denote by $f : R \longrightarrow R$ the Frobenius homomorphism, for which $f(r) = r^p$ for all $r \in R$. We shall use the skew polynomial ring $R[T, f]$ associated to R and f in the indeterminate T over R. Recall that $R[T, f]$ is, as a left R-module, freely generated by $(T^i)_{i\in\mathbb{N}}$, and so consists of all polynomials $\sum_{i=0}^{n} r_iT^i$, where $n \in \mathbb{N}$ and $r_0, \ldots, r_n \in R$; however, its multiplication is subject to the rule

$$Tr = f(r)T = r^pT \quad \text{for all } r \in R.$$

We refer to $R[T, f]$ as the *Frobenius skew polynomial ring associated to R.*

If G is a left $R[T, f]$-module, then the set

$$\Gamma_T(G) := \big\{g \in G : T^jg = 0 \text{ for some } j \in \mathbb{N}_+\big\}$$

is an $R[T, f]$-submodule of G, called the *T-torsion submodule* of G; we say that G is *T-torsion* precisely when $G = \Gamma_T(G)$.

Note that R itself has a natural structure as a left $R[T, f]$-module under which $Tr = f(r)$ for all $r \in R$ (that is, in which the action of the indeterminate T on an element of R is just the same as the action of the Frobenius homomorphism). We shall use ${}_{R[T,f]}\mathrm{Mod}$ to denote the category of all left $R[T, f]$-modules and $R[T, f]$-homomorphisms between them. The category of all R-modules will be denoted by Mod_R.

2. Modules of generalized fractions

The concept of module of generalized fractions (due to Sharp and H. Zakeri [21]) will be used in this paper. The construction and basic properties of these modules can be found in [21], but, at the request of the referee, we include in this section explanation of some of the main ideas.

Reminder 2.1 (R. Y. Sharp and H. Zakeri [21, §2]). Let $k \in \mathbb{N}_+$. Let U be a *triangular subset* of R^k [21, 2.1], that is, a non-empty subset of R^k such that

(i) whenever $(u_1, \ldots, u_k) \in U$ and $n_1, \ldots, n_k \in \mathbb{N}_+$, then $(u_1^{n_1}, \ldots, u_k^{n_k}) \in U$ also; and

(ii) whenever $(u_1, \ldots, u_k), (v_1, \ldots, v_k) \in U$, then there exists $(w_1, \ldots, w_k) \in U$ such that $w_i \in \left(\sum_{j=1}^{i} u_j R\right) \cap \left(\sum_{j=1}^{i} v_j R\right)$ for all $i = 1, \ldots k$, so that there exist $k \times k$ lower triangular matrices $\mathbf{H}$ and $\mathbf{K}$ with entries in R such that

$$\mathbf{H}[u_1, \ldots, u_k]^T = [w_1, \ldots, w_k]^T = \mathbf{K}[v_1, \ldots, v_k]^T.$$

(Here, ^{T} denotes matrix transpose, and $[z_1, \ldots, z_k]^T$ (for $z_1, \ldots, z_k \in R$) is to be interpreted as a $k \times 1$ column matrix in the obvious way.)

It will be convenient for us to use $D_k(R)$ to denote the set of $k \times k$ lower triangular matrices with entries in R; we use $\det(\mathbf{H})$ to denote the determinant of an $\mathbf{H} \in D_k(R)$.

Let M be an R-module. Define a relation $\sim$ on $M \times U$ as follows: for $m, n \in M$ and $(u_1, \ldots, u_k), (v_1, \ldots, v_k) \in U$, write $(m, (u_1, \ldots, u_k)) \sim (n, (v_1, \ldots, v_k))$ precisely when there exist $(w_1, \ldots, w_k) \in U$ and $\mathbf{H}, \mathbf{K} \in D_k(R)$ such that

$$\mathbf{H}[u_1, \ldots, u_k]^T = [w_1, \ldots, w_k]^T = \mathbf{K}[v_1, \ldots, v_k]^T$$

and $\det(\mathbf{H})m - \det(\mathbf{K})n \in \sum_{j=1}^{k-1} w_j M$.

Then $\sim$ is an equivalence relation; for $m \in M$ and $(u_1, \ldots, u_k) \in U$, we denote the equivalence class of $(m, (u_1, \ldots, u_k))$ by the 'generalized fraction'

$$\frac{m}{(u_1, \ldots, u_k)}.$$

The set of all equivalence classes of $\sim$ is an R-module, called the *module of generalized fractions of M with respect to U*, under operations for which, for $m, n \in M$ and $(u_1, \ldots, u_k), (v_1, \ldots, v_k) \in U$,

$$\frac{m}{(u_1, \ldots, u_k)} + \frac{n}{(v_1, \ldots, v_k)} = \frac{\det(\mathbf{H})m + \det(\mathbf{K})n}{(w_1, \ldots, w_k)}$$

for *any* choice of $(w_1, \ldots, w_k) \in U$ and $\mathbf{H}, \mathbf{K} \in D_k(R)$ such that

$$\mathbf{H}[u_1, \ldots, u_k]^T = [w_1, \ldots, w_k]^T = \mathbf{K}[v_1, \ldots, v_k]^T,$$

and, for $r \in R$,

$$r\frac{m}{(u_1, \ldots, u_k)} = \frac{rm}{(u_1, \ldots, u_k)}.$$

This module of generalized fractions is denoted by $U^{-k}M$.

Remark 2.2. Use the notation of 2.1. It is worth bearing in mind, when one is calculating with generalized fractions in $U^{-k}M$, that, whenever $(u_1, \ldots, u_k), (y_1, \ldots, y_k) \in U$ and $\mathbf{L} \in D_k(R)$ are such that $\mathbf{L}[u_1, \ldots, u_k]^T = [y_1, \ldots, y_k]^T$, then, for all $m \in M$,

$$\frac{m}{(u_1, \ldots, u_k)} = \frac{\det(\mathbf{L})m}{(y_1, \ldots, y_k)} \quad \text{in } U^{-k}M.$$

This sometimes permits us to change a denominator to one that is, in some sense, more convenient.

Example 2.3. For our given sequence $\underline{x} = x_1, x_2, \ldots, x_l$ and any $i \in \mathbb{N}_+$, we set

$$U(\underline{x})_i := \{(x_1^{n_1}, \ldots, x_i^{n_i}) : \text{for some } j \in [0, i], n_1, \ldots, n_j \in \mathbb{N}_+, n_{j+1} = \cdots = n_i = 0\},$$

where x_k is interpreted as 1 when $k \geq l + 1$. Then $U(\underline{x})_i$ is a triangular subset of R^i for each $i \in \mathbb{N}_+$.

Discussion 2.4. We use the notation of 2.1.

(i) The construction of a module of generalized fractions can be viewed as a generalization of ordinary fraction formation in commutative algebra: see [21, 3.1].

(ii) For $r_1, \ldots, r_k \in R$, we shall denote the diagonal $k \times k$ matrix in $D_k(R)$ whose diagonal entries are $r_1, \ldots, r_k$ by $\mathbf{diag}(r_1, \ldots, r_k)$. Notice that

$$\det(\mathbf{diag}(r_1, \ldots, r_k)) = r_1 \cdots r_k,$$

and that, for non-negative integers $n_1, \ldots, n_k, m_1, \ldots, m_k$, we have

$$\mathbf{diag}(r_1^{m_1}, \ldots, r_k^{m_k})[r_1^{n_1}, \ldots, r_k^{n_k}]^T = [r_1^{m_1+n_1}, \ldots, r_k^{m_k+n_k}]^T.$$

(iii) The comments in (ii) can be useful. For example, if $(u_1, \ldots, u_k), (w_1, \ldots, w_k) \in U$ are such that there exist $\mathbf{H}, \mathbf{K} \in D_k(R)$ with

$$\mathbf{H}[u_1, \ldots, u_k]^T = [w_1, \ldots, w_k]^T = \mathbf{K}[u_1, \ldots, u_k]^T,$$

then

$$\mathbf{DH}[u_1, \ldots, u_k]^T = [w_1^2, \ldots, w_k^2]^T = \mathbf{DK}[u_1, \ldots, u_k]^T,$$

where $\mathbf{D} := \mathbf{diag}(w_1, \ldots, w_k)$, and it turns out (see [21, Lemma 2.3]) that $\det(\mathbf{DH}) - \det(\mathbf{DK}) \in \sum_{i=1}^{k-1} w_i^2 R$. This can be helpful when one has rather little information about $\mathbf{H}$ but full knowledge of $\mathbf{K}$.

To illustrate this, suppose that it is known, for some $r \in R$, that $\det(\mathbf{H})r \in \sum_{i=1}^{k-1} w_i R$; then $\det(\mathbf{DH})r = w_1 \cdots w_k \det(\mathbf{H})r \in \sum_{i=1}^{k-1} w_i^2 R$, and the above considerations show that

$$\det(\mathbf{DK})r = \det(\mathbf{DH})r - (\det(\mathbf{DH}) - \det(\mathbf{DK}))\, r \in \textstyle\sum_{i=1}^{k-1} w_i^2 R.$$

(iv) The equivalence relation $\sim$ of 2.1 is such that, whenever $(u_1, \ldots, u_k) \in U$ and the element m of M actually belongs to $\sum_{i=1}^{k-1} u_i M$, then

$$\frac{m}{(u_1, \ldots, u_k)} = 0 \quad \text{in } U^{-k}M.$$

(v) Note that, by (iv), if $k \geq 2$ and $(v_1, \ldots, v_{k-2}, 1, 1) \in U$, then

$$\frac{n}{(v_1, \ldots, v_{k-2}, 1, 1)} = 0 \quad \text{in } U^{-k}M, \text{ for all } n \in M.$$

(vi) In the case where U consists entirely of possibly improper regular sequences on M, there is the following converse of (iv), proved by Sharp and Zakeri in [23, Theorem 3.15]: if $(u_1, \dots, u_k) \in U$ and $m \in M$ are such that

$$\frac{m}{(u_1, \dots, u_k)} = 0,$$

then $m \in \sum_{i=1}^{k-1} u_i M$.

Discussion 2.5. A *chain of triangular subsets on* R is a family $\mathcal{U} := (U_i)_{i=1}^{\infty}$ such that

(i) U_i is a triangular subset of R^i for all $i \in \mathbb{N}_+$;
(ii) whenever $(u_1, \dots, u_i) \in U_i$ (for any $i \in \mathbb{N}_+$), then $(u_1, \dots, u_i, 1) \in U_{i+1}$;
(iii) whenever $(u_1, \dots, u_i) \in U_i$ with $i > 1$, then $(u_1, \dots, u_{i-1}) \in U_{i-1}$; and
(iv) $(1) \in U_1$.

Given such a family $\mathcal{U}$ and an R-module M, we can construct a complex $\mathcal{C}(\mathcal{U}, M)$ of modules of generalized fractions

$$0 \xrightarrow{e^{-1}} M \xrightarrow{e^0} U_1^{-1}M \xrightarrow{e^1} \cdots \xrightarrow{e^{i-1}} U_i^{-i}M \xrightarrow{e^i} U_{i+1}^{-i-1}M \to \cdots,$$

in which $e^0(m) = \frac{m}{(1)} \in U_1^{-1}M$ for all $m \in M$ and, for $i \in \mathbb{N}_+$,

$$e^i\left(\frac{m}{(u_1, \dots, u_i)}\right) = \frac{m}{(u_1, \dots, u_i, 1)} \in U_{i+1}^{-i-1}M$$

for all $m \in M$ and $(u_1, \dots, u_i) \in U_i$. The comment in 2.4(v) shows that $\mathcal{C}(\mathcal{U}, M)$ is indeed a complex.

The Exactness Theorem of Sharp–Zakeri ([23, Theorem 3.3], but see L. O'Carroll [15, Theorem 3.1] for a subsequent shorter proof) states that the complex $\mathcal{C}(\mathcal{U}, M)$ is exact if and only if, for all $i \in \mathbb{N}_+$, each element of U_i is a possibly improper regular sequence on M.

Example 2.6. For our sequence $\underline{x} = x_1, x_2, \dots, x_l$, the family $\mathcal{U}(\underline{x}) := (U(\underline{x})_i)_{i=1}^{\infty}$, where the $U(\underline{x})_i$ ($i \in \mathbb{N}_+$) are as defined in 2.3, is a chain of triangular subsets on R. In particular, we can construct the complex of modules of generalized fractions $\mathcal{C}(\mathcal{U}(\underline{x}), R)$, which we shall write as

$$0 \xrightarrow{e^{-1}} R \xrightarrow{e^0} U(\underline{x})_1^{-1}R \xrightarrow{e^1} \cdots \xrightarrow{e^{i-1}} U(\underline{x})_i^{-i}R \xrightarrow{e^i} U(\underline{x})_{i+1}^{-i-1}R \to \cdots.$$

Note that $U(\underline{x})_i^{-i}R = 0$ whenever $i \geq l+2$. When working with a cohomology module $H^i(\mathcal{C}(\mathcal{U}(\underline{x}), R)) = \operatorname{Ker} e^i / \operatorname{Im} e^{i-1}$ (where $i \in \mathbb{N}$) of the complex $\mathcal{C}(\mathcal{U}(\underline{x}), R)$, we shall use '[ ]' to denote natural images, in this cohomology module, of elements of $\operatorname{Ker} e^i$.

In the special case in which $(R, \mathfrak{m})$ is local, $l = \dim R = t$ and $x_1, \dots, x_l$ is a system of parameters of R, it follows from the Exactness Theorem mentioned in 2.5 that R is Cohen–Macaulay if and only if $\mathcal{C}(\mathcal{U}(\underline{x}), R)$ is exact. In [24, Theorems 2.4 and 2.5], Sharp and Zakeri proved that $(R, \mathfrak{m})$ is a generalized Cohen–Macaulay local ring if and only if there is a power of $\mathfrak{m}$ that annihilates all the cohomology modules of the complex $\mathcal{C}(\mathcal{U}(\underline{x}), R)$, and that when this is the case, the ith cohomology module $H^i(\mathcal{C}(\mathcal{U}(\underline{x}), R))$ of the complex $\mathcal{C}(\mathcal{U}(\underline{x}), R)$ is isomorphic to $H^i_{\mathfrak{m}}(R)$ for all $i = 0, \dots, \dim R - 1 = l - 1$. The latter result is relevant to this paper, although in §4 we provide a refinement for use in the case where R has prime characteristic p.

Some of the applications of modules of generalized fractions can be found in [22], [23], [24], [15], [5] and [12]. This paper provides some more.

3. Background results

In this section, we shall collect together some observations and results, with appropriate references, that concern topics mentioned in §1 and which we plan to use.

Remark 3.1. Suppose that $\underline{x} = x_1, \ldots, x_l$ is an unconditioned strong d-sequence in R.

(i) It is immediate from the definitions that, for any $n_1, \ldots, n_l \in \mathbb{N}_+$, $\Lambda \subsetneq [1,l]$ and $j \in [1,l] \setminus \Lambda$, we have

$$\big((\textstyle\sum_{i\in\Lambda} x_i^{n_i}R) : x_j^{n_j}\big) = \big((\textstyle\sum_{i\in\Lambda} x_i^{n_i}R) : x_j\big) = \big(\textstyle\sum_{i\in\Lambda} x_i^{n_i}R\big)^{(\underline{x})\text{-unm}}.$$

(ii) Therefore, if $\mathfrak{a}$ is an ideal of R such that $\mathfrak{a} \subseteq \sqrt{\sum_{i=1}^{l} x_iR}$, then every permutation of $x_1, \ldots, x_l$ is an $\mathfrak{a}$-filter regular sequence, because $\sum_{i=1}^{l} x_iR$ annihilates $\big(\sum_{i\in\Lambda} x_iR : \sum_{i\in[1,l]\setminus\Lambda} x_iR\big)/\big(\sum_{i\in\Lambda} x_iR\big)$ for all $\Lambda \subsetneq [1,l]$.

Theorem 3.2 (P. Schenzel [17, Satz 2.4.2])**.** *Suppose that $(R,\mathfrak{m})$ is local; recall that $\dim(R) = t$. Then, for all systems of parameters $a_1, \ldots, a_t$ of R, the ideal $\prod_{i=0}^{t-1} \mathrm{Ann}_R(H^i_{\mathfrak{m}}(R))$ of R annihilates all the R-modules*

$$\big((\textstyle\sum_{i=1}^{j} a_iR) : a_{j+1}\big)/\big(\textstyle\sum_{i=1}^{j} a_iR\big) \quad (j = 0, \ldots, t-1).$$

In [3], S. Goto and T. Ogawa proved that, in a generalized Cohen–Macaulay local ring $(R,\mathfrak{m})$, there exists a positive integer h such that every system of parameters of R contained in $\mathfrak{m}^h$ is a d-sequence. The following corollary (of Schenzel's Theorem 3.2) is a variation on that theme.

Corollary 3.3. *Suppose that $(R,\mathfrak{m})$ is local with $\dim(R) = t$, and that $\underline{x} = x_1, \ldots, x_l$ is a subsystem of parameters of R such that $\sum_{i=1}^{l} x_iR \subseteq \prod_{i=0}^{t-1} \mathrm{Ann}_R(H^i_{\mathfrak{m}}(R))$. Then $\underline{x}$ is an unconditioned strong d-sequence.*

Proof. Let $n_1, \ldots, n_l$ be positive integers, and let $j \in \mathbb{N}$ be such that $0 \le j < l$. By Schenzel's Theorem 3.2, the R-module $\big((\sum_{i=1}^{j} x_i^{n_i}R) : x_{j+1}^{n_{j+1}}\big)/\big(\sum_{i=1}^{j} x_i^{n_i}R\big)$ is annihilated by $\sum_{i=1}^{l} x_iR$ and so, in particular, by x_{j+1}. Hence

$$\big((\textstyle\sum_{i=1}^{j} x_i^{n_i}R) : x_{j+1}^{n_{j+1}}\big) = \big((\textstyle\sum_{i=1}^{j} x_i^{n_i}R) : x_{j+1}\big).$$

Moreover, the hypotheses on $x_1, \ldots, x_l$ do not depend on the order in which this sequence is written.

Therefore, if j, k are such that $0 \le j < k \le l$, then the fact that x_k annihilates

$$\big((\textstyle\sum_{i=1}^{j} x_i^{n_i}R) : x_{j+1}^{n_{j+1}}\big)/\big(\textstyle\sum_{i=1}^{j} x_i^{n_i}R\big)$$

ensures that $\big((\sum_{i=1}^{j} x_i^{n_i}R) : x_{j+1}^{n_{j+1}}x_k^{n_k}\big) \subseteq \big((\sum_{i=1}^{j} x_i^{n_i}R) : x_k^{n_k+1}\big)$, and the preceding paragraph shows that

$$\big((\textstyle\sum_{i=1}^{j} x_i^{n_i}R) : x_k^{n_k+1}\big) = \big((\textstyle\sum_{i=1}^{j} x_i^{n_i}R) : x_k^{n_k}\big).$$

Therefore $x_1^{n_1}, \ldots, x_l^{n_l}$ is a d-sequence. Since the hypotheses on $x_1, \ldots, x_l$ do not depend on the order in which this sequence is written, we see that $\underline{x}$ is an unconditioned strong d-sequence. □

The next proposition is an extension of a well-known result. For an explanation of what it means to say that a local ring is formally catenary, and for a proof of L. J. Ratliff's Theorem that a universally catenary local ring is formally catenary, the reader is referred to [14, p. 252].

Proposition 3.4. *Assume that* $(R, \mathfrak{m})$ *is a formally catenary local ring all of whose formal fibres are Cohen–Macaulay. (These hypotheses would be satisfied if* R *was an excellent local ring.)*

Suppose that R *is equidimensional and Cohen–Macaulay on the punctured spectrum. Then* R *is a generalized Cohen–Macaulay local ring.*

Proof. Since R is equidimensional and formally catenary, $\widehat{R}$ is equidimensional; the hypothesis concerning the formal fibres ensures that $\widehat{R}$ is Cohen–Macaulay on the punctured spectrum. Thus one can assume that R is complete, and in that case the claim follows from work of P. Schenzel, N. V. Trung and N. T. Cuòng [18, (2.5) and (3.8)]. □

The next two results are entirely due to S. Goto and K. Yamagishi [4]. Unfortunately, as far as we are aware, [4] only exists as a preprint that has been circulating informally for more than 15 years, without formal publication. T. Kawasaki included a proof of [4, Lemma 2.2] in [11, Theorem A.1]; as we have not been able to find a formally published proof of [4, Theorem 2.3], we have included one below.

Theorem 3.5 (S. Goto and K. Yamagishi [4, Lemma 2.2])**.** (See also [11, Theorem A.1].) *Suppose that* $\underline{x} = x_1, \ldots, x_l$ *is an unconditioned strong d-sequence in* R*. Then, for each* $\Delta \subsetneq [1, l]$ *and each* $j \in [1, l] \setminus \Delta$*, and for all positive integers* $n_1, \ldots, n_l$*, we have*

$$\begin{aligned}\big(\textstyle\sum_{i\in\Delta} x_i^{n_i} R\big)^{(\underline{x})\text{-unm}} &= \big(\big(\textstyle\sum_{i\in\Delta} x_i^{n_i} R\big) : x_j\big)\\ &= \textstyle\sum_{\Lambda\subseteq\Delta} \big(\prod_{i\in\Lambda} x_i^{n_i-1}\big)\big(\sum_{i\in\Lambda} x_i R\big)^{(\underline{x})\text{-unm}}.\end{aligned}$$

Proof. This result, originally due to Goto and Yamagishi, is proved in [11, Theorem A.1] in the case where $n_1, \ldots, n_l \geq 2$, and one can check that that proof works for all choices of positive integers $n_1, \ldots, n_l$. □

Theorem 3.6 (S. Goto and K. Yamagishi [4, Theorem 2.3])**.** *Suppose that* $\underline{x} = x_1, \ldots, x_l$ *is an unconditioned strong d-sequence in* R *and let* $n_1, n_2, \ldots, n_l \in \mathbb{N}_+$ *be any positive integers.*

(i) *When* $l = 1$,

$$(x_1^{n_1} R)^{\lim} = x_1^{n_1} R + \textstyle\bigcup_{j\in\mathbb{N}} (0 : x_1^j) = x_1^{n_1} R + (0 : x_1).$$

(ii) *When* $l \geq 2$,

$$\begin{aligned}\big(\textstyle\sum_{i=1}^{l} x_i^{n_i} R\big)^{\lim} &= \textstyle\sum_{i=1}^{l} \big(\big(\sum_{j\in[1,l]\setminus\{i\}} x_j^{n_j} R\big) :_R x_i\big)\\ &= \textstyle\sum_{i=1}^{l} \big(\sum_{j\in[1,l]\setminus\{i\}} x_j^{n_j} R\big)^{(\underline{x})\text{-unm}}.\end{aligned}$$

Proof. (Goto–Yamagishi.) We use induction on l; the result is easy when $l = 1$, and so we suppose that $l \geq 2$ and that the result has been proved for smaller values of l.

Let $a \in \big(\sum_{i=1}^{l} x_i^{n_i} R\big)^{\lim}$, so that, without loss of generality, there exists $m \in \mathbb{N}_+$ such that $x_{[1,l]}^m a \in \sum_{i=1}^{l} x_i^{n_i+m} R$. Thus there exist $b \in \sum_{i=1}^{l-1} x_i^{n_i+m} R$ and $c \in R$ such that $x_{[1,l]}^m a = b + x_l^{n_l+m} c$, and so

$$x_{[1,l-1]}^m a - x_l^{n_l} c \in \big(\big(\textstyle\sum_{i=1}^{l-1} x_i^{n_i+m} R\big) : x_l^m\big) = \big(\big(\textstyle\sum_{i=1}^{l-1} x_i^{n_i+m} R\big) : x_l\big).$$

Therefore, by 3.5, we have

$$(\ddagger) \qquad x_{[1,l-1]}^m a - x_l^{n_l} c = \textstyle\sum_{\Lambda \subseteq [1,l-1]} \prod_{i \in \Lambda} x_i^{n_i+m-1} h_\Lambda,$$

where $h_\Lambda \in \big(\sum_{i \in \Lambda} x_i R\big)^{(\underline{x})\text{-unm}}$ for all $\Lambda \subseteq [1, l-1]$. However, for each $\Lambda \subsetneq [1, l-1]$ and $j \in [1, l-1] \setminus \Lambda$, we have

$$\big(\big(\textstyle\sum_{i \in \Lambda} x_i R\big) : x_l\big) = \big(\textstyle\sum_{i \in \Lambda} x_i R\big)^{(\underline{x})\text{-unm}} = \big(\big(\textstyle\sum_{i \in \Lambda} x_i R\big) : x_j\big)$$

by 3.1(i), so that $x_{[1,l-1]} h_\Lambda \in \sum_{i \in \Lambda} x_i^2 R$. We now multiply both sides of equation (‡) by $x_{[1,l-1]}$ to obtain that

$$x_{[1,l-1]}^{m+1} \Big(a - \textstyle\prod_{i=1}^{l-1} x_i^{n_i-1} h_{[1,l-1]}\Big) \in \textstyle\sum_{i=1}^{l-1} x_i^{n_i+m+1} R + x_l^{n_l} R.$$

Since the natural images of $x_1, \ldots, x_{l-1}$ in the ring $R/x_l^{n_l} R$ form an unconditioned strong d-sequence in that ring, it follows from the inductive hypothesis that

$$\begin{aligned} a - \textstyle\prod_{i=1}^{l-1} x_i^{n_i-1} h_{[1,l-1]} &\in \textstyle\sum_{i=1}^{l-1} \big(\big(\sum_{j \in [1,l-1] \setminus \{i\}} x_j^{n_j} R + x_l^{n_l} R\big) :_R x_i\big) + x_l^{n_l} R + x_1^{n_1} R \\ &= \textstyle\sum_{i=1}^{l-1} \big(\big(\sum_{j \in [1,l-1] \setminus \{i\}} x_j^{n_j} R + x_l^{n_l} R\big) :_R x_i\big) + x_1^{n_1} R. \end{aligned}$$

(The presence of the ideal $x_1^{n_1} R$ on the right hand side ensures that the argument applies to the case where $l = 2$.) Since $\big(\sum_{i=1}^{l-1} x_i R\big)^{(\underline{x})\text{-unm}} = \big(\big(\sum_{i=1}^{l-1} x_i R\big) : x_l\big)$, we see that

$$\textstyle\prod_{i=1}^{l-1} x_i^{n_i-1} h_{[1,l-1]} \in \big(\big(\sum_{j=1}^{l-1} x_j^{n_j} R\big) :_R x_l\big)$$

(even in the case where $l = 2$) and so $a \in \sum_{i=1}^{l} \big(\big(\sum_{j \in [1,l] \setminus \{i\}} x_j^{n_j} R\big) :_R x_i\big)$. Therefore

$$\big(\textstyle\sum_{i=1}^{l} x_i^{n_i} R\big)^{\lim} \subseteq \textstyle\sum_{i=1}^{l} \big(\big(\sum_{j \in [1,l] \setminus \{i\}} x_j^{n_j} R\big) :_R x_i\big) = \textstyle\sum_{i=1}^{l} \big(\sum_{j \in [1,l] \setminus \{i\}} x_j^{n_j} R\big)^{(\underline{x})\text{-unm}},$$

and the reverse inclusion is easy. □

The following corollary is immediate from 3.5 and 3.6.

Corollary 3.7. *Suppose that $\underline{x} = x_1, \ldots, x_l$ is an unconditioned strong d-sequence in R (with $l \geq 1$) and let $n_1, n_2, \ldots, n_l \in \mathbb{N}_+$ be any positive integers. Then*

(i) $x_{[1,l]} \big(\sum_{i=1}^{l} x_i^{n_i} R\big)^{\lim} \subseteq \sum_{i=1}^{l} x_i^{n_i+1} R$; *and*

(ii) $\big(\sum_{i=1}^{l} x_i^{n_i} R\big)^{\lim} = \sum_{\Lambda \subsetneq [1,l]} \big(\prod_{i \in \Lambda} x_i^{n_i-1}\big) \big(\sum_{i \in \Lambda} x_i R\big)^{(\underline{x})\text{-unm}}$ *when* $l \geq 2$.

Theorem 3.8. *Let $\underline{x} = x_1, x_2, \ldots, x_l$ be a d-sequence in R.*

(i) (C. Huneke [9, Proposition 2.1].) *Then*

$$\big(\big(\textstyle\sum_{i=1}^{r} x_i R\big) : x_{r+1}\big) \bigcap \big(\textstyle\sum_{i=1}^{l} x_i R\big) = \textstyle\sum_{i=1}^{r} x_i R \quad \text{for all } r = 0, \ldots, l-1.$$

(ii) *If $\underline{x} = x_1, x_2, \ldots, x_l$ is an unconditioned strong d-sequence in R, then*

$$\big(\textstyle\sum_{i \in \Lambda} x_i^{n_i} R\big)^{(\underline{x})\text{-unm}} \bigcap \big(\textstyle\sum_{i=1}^{l} x_i^{n_i} R\big) = \textstyle\sum_{i \in \Lambda} x_i^{n_i} R$$

for all positive integers $n_1, n_2, \ldots, n_l$ and each $\Lambda \subsetneq [1, l]$.

Proof. (ii) This is immediate from Huneke's result quoted in part (i), because

$$\big(\sum_{i\in\Lambda} x_i^{n_i}R\big)^{(\underline{x})\text{-unm}} = \big(\big(\sum_{i\in\Lambda} x_i^{n_i}R\big) : x_j\big) = \big(\big(\sum_{i\in\Lambda} x_i^{n_i}R\big) : x_j^{n_j}\big) \quad \text{for } j \in [1,l] \setminus \Lambda:$$

see 3.1(i). □

As in [10], use will be made of the following extension, due to G. Lyubeznik, of a result of R. Hartshorne and R. Speiser. It shows that, when R is local and of prime characteristic p, a T-torsion left $R[T,f]$-module which is Artinian (that is, 'cofinite' in the terminology of Hartshorne and Speiser) as an R-module exhibits a certain uniformity of behaviour.

Theorem 3.9 (G. Lyubeznik [13, Proposition 4.4])**.** (Compare Hartshorne–Speiser [6, Proposition 1.11].) *Suppose that $(R,\mathfrak{m})$ is local and of prime characteristic p, and let G be a left $R[T,f]$-module which is Artinian as an R-module. Then there exists $e \in \mathbb{N}$ such that $T^e\Gamma_T(G) = 0$.*

Hartshorne and Speiser first proved this result in the case where R is local and contains its residue field which is perfect. Lyubeznik applied his theory of F-modules to obtain the result without restriction on the local ring R of characteristic p. A short proof of the theorem, in the generality achieved by Lyubeznik, is provided in [20].

Lemma 3.10 (Katzman–Sharp [10, Lemma 3.5])**.** *Suppose that R has prime characteristic p.*

(i) *Let $n \in \mathbb{N}_+$ and let U be a triangular subset of R^n. Then the module of generalized fractions $U^{-n}R$ has a structure as left $R[T,f]$-module with*

$$T\left(\frac{r}{(u_1,\ldots,u_n)}\right) = \frac{r^p}{(u_1^p,\ldots,u_n^p)} \quad \textit{for all } r \in R \textit{ and } (u_1,\ldots,u_n) \in U.$$

(ii) *It follows easily that the complex $\mathcal{C}(\mathcal{U}(\underline{x}),R)$ of modules of generalized fractions of 2.6 is a complex of left $R[T,f]$-modules and $R[T,f]$-homomorphisms; hence all its cohomology modules $H^i(\mathcal{C}(\mathcal{U}(\underline{x}),R))$ $(i \in \mathbb{N})$ have natural structures as left $R[T,f]$-modules.*

4. Preparatory results

Most of the results in this section concern the case where R has prime characteristic p, but the first two do not.

Lemma 4.1. *Suppose that $(R,\mathfrak{m})$ is local, and consider the complex of modules of generalized fractions $\mathcal{C}(\mathcal{U}(\underline{x}),R)$ of 2.6. Let r be an integer such that $0 \le r < l$. (In the case where $r = 0$, a generalized fraction such as $h/(x_1,\ldots,x_r)$ (where $h \in R$) is to be interpreted simply as h.)*

(i) *If $h \in \big(\big(\sum_{i=1}^r x_iR\big) : x_{r+1}\big)$, then*

$$\frac{h}{(x_1,\ldots,x_r)} \in \operatorname{Ker} e^r, \quad \textit{so that} \quad \left[\frac{h}{(x_1,\ldots,x_r)}\right] \in H^r(\mathcal{C}(\mathcal{U}(\underline{x}),R)).$$

(The notation '[ ]' is explained in 2.6.)

(ii) *Let* $n \in \mathbb{N}_+$ *and* $h \in R$. *Then*

$$\frac{h}{(x_1^n, \ldots, x_r^n)} \in \operatorname{Im} e^{r-1}$$

if and only if $h \in \left(\sum_{i=1}^r x_i^n R\right)^{\lim}$.

Proof. When $r = 0$, all the claims are easy. We therefore omit the proofs in that case and assume that $1 \le r < l$.

(i) Since $x_{r+1}h \in \sum_{i=1}^r x_i R$, it is immediate from 2.2 and 2.4(iv) that

$$e^r\left(\frac{h}{(x_1, \ldots, x_r)}\right) = \frac{h}{(x_1, \ldots, x_r, 1)} = \frac{x_{r+1}h}{(x_1, \ldots, x_r, x_{r+1})} = 0 \in U(\underline{x})_{r+1}^{-r-1}R.$$

(ii) ($\Leftarrow$) Assume that $h \in \left(\sum_{i=1}^r x_i^n R\right)^{\lim}$. Thus there exists $m \in \mathbb{N}$ such that $x_{[1,r]}^m h \in \sum_{i=1}^r x_i^{n+m} R$. Thus we can write $x_{[1,r]}^m h = \sum_{i=1}^r s_i x_i^{n+m}$ for some $s_1, \ldots, s_r \in R$. Then, in $U(\underline{x})_r^{-r}R$, we have

$$\begin{aligned}
\frac{h}{(x_1^n, \ldots, x_{r-1}^n, x_r^n)} &= \frac{x_{[1,r]}^m h}{(x_1^{n+m}, \ldots, x_{r-1}^{n+m}, x_r^{n+m})} = \frac{\sum_{i=1}^r s_i x_i^{n+m}}{(x_1^{n+m}, \ldots, x_{r-1}^{n+m}, x_r^{n+m})} \\
&= \frac{s_r x_r^{n+m}}{(x_1^{n+m}, \ldots, x_{r-1}^{n+m}, x_r^{n+m})} \qquad \text{(on use of 2.4(iv))} \\
&= \frac{s_r}{(x_1^{n+m}, \ldots, x_{r-1}^{n+m}, 1)} \\
&= \begin{cases} e^{r-1}(s_r) & \text{if } r = 1, \\ e^{r-1}\left(\frac{s_r}{(x_1^{n+m}, \ldots, x_{r-1}^{n+m})}\right) & \text{if } r \ge 2. \end{cases}
\end{aligned}$$

($\Rightarrow$) By 2.2, we can write, in $U(\underline{x})_r^{-r}R$,

$$\frac{h}{(x_1^n, \ldots, x_r^n)} = e^{r-1}\left(\frac{g}{(x_1^{m+n}, x_2^{m+n}, \ldots, x_{r-1}^{m+n})}\right) = \frac{x_r^{m+n} g}{(x_1^{m+n}, x_2^{m+n}, \ldots, x_r^{m+n})}$$

for some $m \in \mathbb{N}$ and $g \in R$. Thus

$$\frac{x_{[1,r]}^m h - x_r^{m+n} g}{(x_1^{m+n}, x_2^{m+n}, \ldots, x_r^{m+n})} = 0 \quad \text{in } U(\underline{x})_r^{-r}R.$$

By the definition of modules of generalized fractions (see 2.1), there exist $u \in \mathbb{N}$ and $\mathbf{H} \in D_r(R)$ such that

$$\mathbf{H}[x_1^{m+n}, \ldots, x_r^{m+n}]^T = [x_1^{m+n+u}, \ldots, x_r^{m+n+u}]^T$$

and $\det(\mathbf{H})(x_{[1,r]}^m h - x_r^{m+n} g) \in \sum_{i=1}^{r-1} x_i^{m+n+u} R$. Since

$$\mathbf{diag}(x_1^u, \ldots, x_r^u)[x_1^{m+n}, \ldots, x_r^{m+n}]^T = [x_1^{m+n+u}, \ldots, x_r^{m+n+u}]^T,$$

we can use the method of 2.4(iii) to see that

$$x_{[1,r]}^{m+n+u} x_{[1,r]}^u (x_{[1,r]}^m h - x_r^{m+n} g) \in \sum_{i=1}^{r-1} x_i^{2m+2n+2u} R.$$

Consequently, $x_{[1,r]}^{2m+n+2u} h \in \sum_{i=1}^{r-1} x_i^{2m+2n+2u} R + x_r^{2m+2n+2u} R$; this implies that

$$h \in \left(\left(\sum_{i=1}^r x_i^{2m+2n+2u} R\right) : x_{[1,r]}^{2m+n+2u}\right) \subseteq \left(\sum_{i=1}^r x_i^n R\right)^{\lim},$$

as required. ∎

It is an immediate consequence of Corollary 3.3 that a generalized Cohen–Macaulay local ring has a system of parameters that is an unconditioned strong d-sequence. We now use modules of generalized fractions to establish a converse of this.

Theorem 4.2. *Suppose that $(R, \mathfrak{m})$ is local; recall that $\dim(R) = t$. Then the following statements are equivalent:*

(i) *R is generalized Cohen–Macaulay;*
(ii) *there exists $h \in \mathbb{N}_+$ such that $y_1^k, \ldots, y_t^k$ is an undconditioned strong d-sequence, for every system of parameters $y_1, \ldots, y_t$ of R and every $k \geq h$;*
(iii) *there exists a system of parameters of R which is an unconditioned strong d-sequence.*

Proof. (i) $\Rightarrow$ (ii) This is immediate from Corollary 3.3 and the definition of generalized Cohen–Macaulay local ring: just choose h so that $\mathfrak{m}^h \subseteq \bigcap_{i=0}^{t-1} \operatorname{Ann}_R(H^i_{\mathfrak{m}}(R))$.

(ii) $\Rightarrow$ (iii) This is clear.

(iii) $\Rightarrow$ (i) Take $l = t$ and $x_1, \ldots, x_t$ to be a system of parameters of R that is an unconditioned strong d-sequence. By 3.1(ii), every permutation of $x_1, \ldots, x_t$ is an $\mathfrak{m}$-filter regular sequence. In order to show that R is generalized Cohen–Macaulay, it is enough, by symmetry, to show that $x_{i+1} H^i_{\mathfrak{m}}(R) = 0$ for all $i = 0, \ldots, t-1$.

We now apply [24, Corollary 2.3 and Theorem 2.4] to the complex $\mathcal{C}(\mathcal{U}(\underline{x}), R)$ of 2.6. Note that, for all $i \in \mathbb{N}_+$, every element of $U(\underline{x})_i$ is an $\mathfrak{m}$-filter regular sequence. The cited results from [24] therefore show that

$$H^i(\mathcal{C}(\mathcal{U}(\underline{x}), R)) = \operatorname{Ker} e^i / \operatorname{Im} e^{i-1} \cong H^i_{\mathfrak{m}}(R) \quad \text{for all } i = 0, \ldots, t-1.$$

It is therefore enough for us to show that, for an $i \in \{0, \ldots, t-1\}$, we have $x_{i+1} \operatorname{Ker} e^i \subseteq \operatorname{Im} e^{i-1}$. We shall deal here with the case where $i > 0$, and leave to the reader the (easy) modification for the case where $i = 0$.

Let $\alpha \in \operatorname{Ker} e^i$. By 2.2, we can write

$$\alpha = \frac{r}{(x_1^n, \ldots, x_i^n)} \quad \text{for some } r \in R \text{ and } n \in \mathbb{N}_+.$$

Therefore

$$\frac{x_{i+1}^n r}{(x_1^n, \ldots, x_i^n, x_{i+1}^n)} = \frac{r}{(x_1^n, \ldots, x_i^n, 1)} = e^i(\alpha) = 0.$$

By 2.1 and 2.2, this means that there exist $v \in \mathbb{N}_+$ and $\mathbf{H} \in D_{i+1}(R)$ such that

$$\mathbf{H}[x_1^n, \ldots, x_{i+1}^n]^T = [x_1^{n+v}, \ldots, x_{i+1}^{n+v}]^T \quad \text{and} \quad \det(\mathbf{H}) x_{i+1}^n r \in \textstyle\sum_{j=1}^i x_j^{n+v} R.$$

Since $\mathbf{diag}(x_1^v, \ldots, x_{i+1}^v)[x_1^n, \ldots, x_{i+1}^n]^T = [x_1^{n+v}, \ldots, x_{i+1}^{n+v}]^T$, we can use the technique of 2.4(iii) to see that $x_{[1,i+1]}^{n+2v} x_{i+1}^n r \in \sum_{j=1}^i x_j^{2n+2v} R$. Therefore, since $x_1, \ldots, x_t$ is an unconditioned strong d-sequence,

$$x_{[1,i]}^{n+2v} r \in \left(\textstyle\sum_{j=1}^i x_j^{2n+2v} R : x_{i+1}^{2n+2v} \right) = \left(\textstyle\sum_{j=1}^i x_j^{2n+2v} R : x_{i+1} \right).$$

Hence $x_{i+1} r \in \left(\sum_{j=1}^i x_j^{2n+2v} R : x_{[1,i]}^{n+2v} \right) \subseteq \left(\sum_{j=1}^i x_j^n R \right)^{\lim}$, so that

$$x_{i+1}\alpha = x_{i+1} \frac{r}{(x_1^n, \ldots, x_i^n)} = \frac{x_{i+1} r}{(x_1^n, \ldots, x_i^n)} \in \operatorname{Im} e^{i-1}$$

by Lemma 4.1(ii). □

It is well known that, when R has prime characteristic p, each local cohomology module $H^i_{\mathfrak{a}}(R)$, where $i \in \mathbb{N}$, has a natural structure as a left $R[T, f]$-module. A detailed explanation is given in [10, 2.1], and the argument there can easily be modified to show that, if M is an arbitrary left $R[T, f]$-module, then $H^i_{\mathfrak{a}}(M)$ (formed by regarding M as an R-module by restriction of scalars) inherits a natural structure as a left $R[T, f]$-module. However, in this paper, we are going to use the following rather stronger statement.

Proposition 4.3. *Suppose that R has prime characteristic p. Then $\left(H^i_{\mathfrak{a}}\right)_{i\in\mathbb{N}}$ is a negative strongly connected sequence of covariant functors from ${}_{R[T,f]}\mathrm{Mod}$ to itself.*

Note. We identify $H^0_{\mathfrak{a}}$ with the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ in the natural way. If M is a left $R[T, f]$-module, then $\Gamma_{\mathfrak{a}}(M)$ is an $R[T, f]$-submodule of M. It should be noted from the proof below that this $R[T, f]$-module structure on $\Gamma_{\mathfrak{a}}(M)$ is exactly the same as the natural left $R[T, f]$-module structure on $H^0_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M)$ provided by the proposition.

Proof. As this proof relies on the Independence Theorem for local cohomology (see [2, 4.2.1]), we shall use notation similar to that employed in [2, §4.2]. Let $\lceil\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ denote the functor obtained from restriction of scalars using the Frobenius homomorphism f: thus, if Y is an R-module, then $Y\lceil$ denotes Y considered as an R-module via f.

Let M be a left $R[T, f]$-module. The map $\tau_M : M \longrightarrow M\lceil$ defined by $\tau_M(m) = Tm$ is an R-module homomorphism. Consequently, for each $i \in \mathbb{N}$, there is an induced R-homomorphism $H^i_{\mathfrak{a}}(\tau_M) : H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M\lceil)$.

Let

$$\Theta = (\theta^i)_{i\in\mathbb{N}} : \left(H^i_{\mathfrak{a}}(\,\bullet\,\lceil)\right)_{i\in\mathbb{N}} \xrightarrow{\cong} \left(H^i_{\mathfrak{a}^{[p]}}(\,\bullet\,)\lceil\right)_{i\in\mathbb{N}}$$

be the isomorphism of negative (strongly) connected sequences of covariant functors (from Mod_R to Mod_R) that is inverse to the one given in the Independence Theorem for local cohomology, in the form in which it is stated in [2, 4.2.1]. Thus θ^0 is the identity natural equivalence. Since $\mathfrak{a}$ and $\mathfrak{a}^{[p]}$ have the same radical, $H^i_{\mathfrak{a}}$ and $H^i_{\mathfrak{a}^{[p]}}$ are the same functor, for each $i \in \mathbb{N}$.

Consider the $\mathbb{Z}$-endomorphism $\xi^i_M := \theta^i_M \circ H^i_{\mathfrak{a}}(\tau_M) : H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M)\lceil$. We now modify the argument of [10, 2.1] and use [10, Lemma 1.3] to show that $H^i_{\mathfrak{a}}(M)$ has a natural structure as a left $R[T, f]$-module in which $T\gamma = \xi^i_M(\gamma)$ for all $\gamma \in H^i_{\mathfrak{a}}(M)$.

Furthermore, if $\beta : M \longrightarrow N$ is an $R[T, f]$-homomorphism of left $R[T, f]$-modules, then

$$\begin{array}{ccc} M & \xrightarrow{\tau_M} & M\lceil \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\beta\lceil} \\ N & \xrightarrow{\tau_N} & N\lceil \end{array}$$

is a commutative diagram of R-homomorphisms, so that, for $i \in \mathbb{N}$, the diagram

$$\begin{array}{ccccc}
H^i_{\mathfrak{a}}(M) & \xrightarrow{H^i_{\mathfrak{a}}(\tau_M)} & H^i_{\mathfrak{a}}(M\lceil) & \xrightarrow{\theta^i_M} & H^i_{\mathfrak{a}}(M)\lceil \\
\Big\downarrow {\scriptstyle H^i_{\mathfrak{a}}(\beta)} & & \Big\downarrow {\scriptstyle H^i_{\mathfrak{a}}(\beta\lceil)} & & \Big\downarrow {\scriptstyle H^i_{\mathfrak{a}}(\beta)\lceil} \\
H^i_{\mathfrak{a}}(N) & \xrightarrow{H^i_{\mathfrak{a}}(\tau_N)} & H^i_{\mathfrak{a}}(N\lceil) & \xrightarrow{\theta^i_N} & H^i_{\mathfrak{a}}(N)\lceil
\end{array}$$

also commutes. This means that, when $H^i_{\mathfrak{a}}(M)$ and $H^i_{\mathfrak{a}}(N)$ are given their natural structures as left $R[T,f]$-modules, as in the preceding paragraph, then the R-homomorphism $H^i_{\mathfrak{a}}(\beta) : H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(N)$ is an $R[T,f]$-homomorphism. In this way, $H^i_{\mathfrak{a}}$ becomes a functor from ${}_{R[T,f]}\mathrm{Mod}$ to itself.

Next, whenever $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$ is an exact sequence of left $R[T,f]$-modules and $R[T,f]$-homomorphisms, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \tau_L} & & \Big\downarrow {\scriptstyle \tau_M} & & \Big\downarrow {\scriptstyle \tau_N} & & \\
0 & \longrightarrow & L\lceil & \xrightarrow{\alpha\lceil} & M\lceil & \xrightarrow{\beta\lceil} & N\lceil & \longrightarrow & 0
\end{array}$$

of R-modules and R-homomorphisms commutes, and so the vertical maps induce a morphism of the long exact sequence of local cohomology modules of the upper sequence into that for the lower sequence. It follows from this (and properties of the isomorphism Θ of connected sequences) that the connecting R-homomorphisms

$$H^i_{\mathfrak{a}}(N) \longrightarrow H^{i+1}_{\mathfrak{a}}(L) \quad (i \in \mathbb{N})$$

are all homomorphisms of left $R[T,f]$-modules. Hence the long exact sequence of local cohomology R-modules induced by $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$ is actually a long exact sequence of left $R[T,f]$-modules and $R[T,f]$-homomorphisms.

Everything else needed for completion of the proof is now straightforward. □

Our next result can be viewed as a strengthening, in the particular case where R has prime characteristic p, of special cases of [24, Theorem (2.4)] and of results of K. Khashyarmanesh, Sh. Salarian and H. Zakeri in [12, Theorem 1.2 and Consequences 1.3(i)] (which refer for proof to the proof of [24, Theorem (2.4)]).

Theorem 4.4. *Suppose that $(R, \mathfrak{m})$ is local and of prime characteristic p, and that $\underline{x} = x_1, \ldots, x_l$ is an $\mathfrak{m}$-filter regular sequence of elements of $\mathfrak{m}$. Consider the complex $\mathcal{C}(\mathcal{U}(\underline{x}), R)$ of modules of generalized fractions of* 2.6*, and note that, by* 3.10(ii)*, this is a complex of left $R[T,f]$-modules and $R[T,f]$-homomorphisms. Then there are isomorphisms of $R[T,f]$-modules*

$$H^i(\mathcal{C}(\mathcal{U}(\underline{x}), R)) \cong H^i_{\mathfrak{m}}(R) \quad \textit{for all } i = 0, \ldots, l-1,$$

where the $H^i_{\mathfrak{m}}(R)$ are considered as left $R[T,f]$-modules in the natural way described in 4.3.

Proof. First, it follows from [21, 3.2] and [22, 2.2] that $H^j_{\mathfrak{m}}\left(U(\underline{x})_i^{-i}R\right) = 0$ for all $i = 1, \ldots, l$ and all $j \geq 0$. Second, one can use 1.2(i) and the Exactness Theorem for complexes of modules of generalized fractions (see 2.5) (in conjunction with [5, Proposition 2.1]) to see that $\operatorname{Supp}\left(H^i(\mathcal{C}(\mathcal{U}(\underline{x}), R))\right) \subseteq \{\mathfrak{m}\}$ for all $i \geq 0$.

With these observations, the theorem can be proved by an obvious modification of the argument used to prove [24, Theorem (2.4)], provided one notes that all the sequences

$$0 \longrightarrow \operatorname{Ker} e^0 \longrightarrow R \longrightarrow \operatorname{Im} e^0 \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Ker} e^i \longrightarrow U(\underline{x})_i^{-i}R \longrightarrow \operatorname{Im} e^i \longrightarrow 0 \quad (1 \leq i \leq l),$$

$$0 \longrightarrow \operatorname{Im} e^{i-1} \longrightarrow U(\underline{x})_i^{-i}R \longrightarrow \operatorname{Coker} e^{i-1} \longrightarrow 0 \quad (1 \leq i \leq l)$$

and

$$0 \longrightarrow \operatorname{Im} e^{i-1} \longrightarrow \operatorname{Ker} e^i \longrightarrow \operatorname{Ker} e^i / \operatorname{Im} e^{i-1} \longrightarrow 0 \quad (1 \leq i \leq l)$$

are exact sequences of left $R[T, f]$-modules and $R[T, f]$-homomorphisms, so that, by Proposition 4.3, all the isomorphisms of local cohomology modules that they induce are $R[T, f]$-isomorphisms. □

Corollary 4.5. *Suppose that $(R, \mathfrak{m})$ is local and of prime characteristic p; recall that $\dim(R) = t$. Since the local cohomology modules $H^i_{\mathfrak{m}}(R)$ are left $R[T, f]$-modules that are Artinian as R-modules, it follows from the Hartshorne–Speiser–Lyubeznik Theorem 3.9 that there exists $e_1 \in \mathbb{N}$ such that*

$$T^{e_1}\Gamma_T(H^i_{\mathfrak{m}}(R)) = 0 \quad \text{for all } i = 0, \dots, t-1.$$

Set $Q_1 := p^{e_1}$. Then Q_1 has the following property: whenever $\underline{x} = x_1, x_2, \dots, x_t$ is a system of parameters of R that is also an $\mathfrak{m}$-filter regular sequence, and whenever r is an integer with $0 \leq r < t$ and $h \in \left(\sum_{i=1}^r x_iR\right)^{(\underline{x})\text{-unm}}$ is such that $h^q \in \left(\sum_{i=1}^r x_i^qR\right)^{\lim}$ for some q, then $h^{Q_1} \in \left(\sum_{i=1}^r x_i^{Q_1}R\right)^{\lim}$.

Proof. In the case where $r = 0$, a generalized fraction such as $h/(x_1, \dots, x_r)$ (where $h \in R$) is to be interpreted simply as h. Let $\underline{x} = x_1, x_2, \dots, x_t$, r and h be as in the statement of the corollary.

Consider the complex of modules of generalized fractions $\mathcal{C}(\mathcal{U}(\underline{x}), R)$ of 2.6. Since $h \in \left(\sum_{i=1}^r x_iR\right)^{(\underline{x})\text{-unm}} \subseteq \left(\left(\sum_{i=1}^r x_iR\right) : x_{r+1}\right)$ by 1.1(vii), it follows from Lemma 4.1(i) that

$$\frac{h}{(x_1, \dots, x_r)} \in \operatorname{Ker} e^r, \quad \text{so that } \left[\frac{h}{(x_1, \dots, x_r)}\right] \in H^r(\mathcal{C}(\mathcal{U}(\underline{x}), R)).$$

Since $h^q \in \left(\sum_{i=1}^r x_i^qR\right)^{\lim}$, it follows from Lemma 4.1(ii) that

$$\left[\frac{h}{(x_1, \dots, x_r)}\right] \in \Gamma_T\left(H^r(\mathcal{C}(\mathcal{U}(\underline{x}), R))\right).$$

Now $H^r(\mathcal{C}(\mathcal{U}(\underline{x}), R)) \cong H^r_{\mathfrak{m}}(R)$ as left $R[T, f]$-modules, by Theorem 4.4. Therefore

$$\left[\frac{h^{Q_1}}{(x_1^{Q_1}, \dots, x_r^{Q_1})}\right] = T^{e_1}\left[\frac{h}{(x_1, \dots, x_r)}\right] = 0, \quad \text{so that } \frac{h^{Q_1}}{(x_1^{Q_1}, \dots, x_r^{Q_1})} \in \operatorname{Im} e^{r-1}.$$

Therefore $h^{Q_1} \in \left(\sum_{i=1}^r x_i^{Q_1}R\right)^{\lim}$ by Lemma 4.1(ii). □

Proposition 4.6. *Suppose that* $(R, \mathfrak{m})$ *is local and of prime characteristic* p*; recall that* $\dim(R) = t$*. Then there exists* Q_2 *such that, for each system of parameters* $\underline{x} = x_1, \ldots, x_t$ *of* R*, we have* $\big(\big(\sum_{i=1}^t x_i R\big)^F\big)^{[Q_2]} \subseteq \big(\sum_{i=1}^t x_i^{Q_2} R\big)^F \bigcap \big(\sum_{i=1}^t x_i^{Q_2} R\big)^{\lim}$*. (Note that* $\sum_{i=1}^t x_i^{Q_2} R = \big(\sum_{i=1}^t x_i R\big)^{[Q_2]}$*.)*

Proof. Our intention is to apply the Hartshorne–Speiser–Lyubeznik Theorem 3.9 to the top local cohomology module $H^t_{\mathfrak{m}}(R)$ of R. Recall that $H^t_{\mathfrak{m}}(R)$ can be realized as the t-th cohomology module of the Čech complex of R with respect to $x_1, \ldots, x_t$. Thus $H^t_{\mathfrak{m}}(R)$ can be represented as the residue class module of $R_{x_1 \cdots x_t}$ modulo the image, under the Čech 'differentiation' map, of $\bigoplus_{i=1}^t R_{x_1 \cdots x_{i-1} x_{i+1} \cdots x_t}$. See [2, §5.1]. We use '[ ]' to denote natural images of elements of $R_{x_1 \cdots x_t}$ in this residue class module.

Recall also (from, for example, [10, 2.3]) that the natural left $R[T, f]$-module structure on $H^t_{\mathfrak{m}}(R)$ is such that

$$T\left[\frac{r}{(x_1 \cdots x_t)^n}\right] = \left[\frac{r^p}{(x_1 \cdots x_t)^{np}}\right] \qquad \text{for all } r \in R \text{ and } n \in \mathbb{N}.$$

Since $H^t_{\mathfrak{m}}(R)$ is an Artinian R-module, it follows from the Hartshorne–Speiser–Lyubeznik Theorem 3.9 that there exists $e_2 \in \mathbb{N}$ such that $T^{e_2}\Gamma_T\left(H^t_{\mathfrak{m}}(R)\right) = 0$. Set $Q_2 = p^{e_2}$.

Let $a \in \big(\sum_{i=1}^t x_i R\big)^F$, so that there exists $Q = p^e$ such that $a^Q \in \big(\sum_{i=1}^t x_i R\big)^{[Q]} = \sum_{i=1}^t x_i^Q R$. Thus, in $H^t_{\mathfrak{m}}(R)$, we have

$$T^e\left[\frac{a}{x_1 \cdots x_t}\right] = \left[\frac{a^Q}{(x_1 \cdots x_t)^Q}\right] = 0, \quad \text{so that} \left[\frac{a^{Q_2}}{(x_1 \cdots x_t)^{Q_2}}\right] = T^{e_2}\left[\frac{a}{x_1 \cdots x_t}\right] = 0.$$

By [10, (2.3)(i)], this means that $a^{Q_2} \in \big(\sum_{i=1}^t x_i^{Q_2} R\big)^{\lim}$. It is easy to check that $\big(\big(\sum_{i=1}^t x_i R\big)^F\big)^{[Q_2]} \subseteq \big(\sum_{i=1}^t x_i^{Q_2} R\big)^F$, and so the proof is complete. □

Lemma 4.7. *Suppose that* $(R, \mathfrak{m})$ *is local and of prime characteristic* p*. If the ideal* $\mathfrak{a}$ *of* R *satisfies* $((\mathfrak{a}\widehat{R})^F)^{[Q]} = (\mathfrak{a}\widehat{R})^{[Q]}$*, then* $(\mathfrak{a}^F)^{[Q]} = \mathfrak{a}^{[Q]}$*.*

Proof. Let $r \in \mathfrak{a}^F$; then $r \in (\mathfrak{a}\widehat{R})^F$ in the ring $\widehat{R}$. Thus $r^Q \in (\mathfrak{a}\widehat{R})^{[Q]} \cap R = (\mathfrak{a}^{[Q]}\widehat{R}) \cap R$, and the latter ideal is just $\mathfrak{a}^{[Q]}$ because $\widehat{R}$ is a faithfully flat extension of R. □

Lemma 4.8. *Suppose that* R *is of prime characteristic* p*. If the ideals* $\mathfrak{a}$ *and* $\mathfrak{n}$ *of* R *satisfy* $\mathfrak{n}^{[Q']} = 0$ *and* $(((\mathfrak{a}+\mathfrak{n})/\mathfrak{n})^F)^{[\widetilde{Q}]} = ((\mathfrak{a}+\mathfrak{n})/\mathfrak{n})^{[\widetilde{Q}]}$ *(in* $R/\mathfrak{n}$*), then* $(\mathfrak{a}^F)^{[Q'\widetilde{Q}]} = \mathfrak{a}^{[Q'\widetilde{Q}]}$*.*

Proof. Let $r \in \mathfrak{a}^F$; then $r + \mathfrak{n} \in ((\mathfrak{a} + \mathfrak{n})/\mathfrak{n})^F$ in $R/\mathfrak{n}$. Thus $(r + \mathfrak{n})^{\widetilde{Q}} \in ((\mathfrak{a} + \mathfrak{n})/\mathfrak{n})^{[\widetilde{Q}]}$; that is, $r^{\widetilde{Q}} \in \mathfrak{a}^{[\widetilde{Q}]} + \mathfrak{n}$. Consequently, $r^{Q'\widetilde{Q}} \in \mathfrak{a}^{[Q'\widetilde{Q}]}$. □

5. The main results

Throughout this section, we assume that R has prime characteristic p.

Proposition 5.1. *Suppose that* R *is of prime characteristic* p*. Assume that* R *is semi-local or that the integral closure of* $R/\sqrt{0}$ *in its total ring of fractions is module-finite over* $R/\sqrt{0}$ *(this would be the case if* R *was excellent). Then there exists* Q_3 *such that* $((xR)^F)^{[Q_3]} = (xR)^{[Q_3]}$ *for all* $x \in R^\circ := R \setminus \bigcup_{\mathfrak{p} \in \min(R)} \mathfrak{p}$*.*

Proof. In case R is semi-local, as everything involved commutes with localization at the finitely many maximal ideals of R, we can assume that $(R, \mathfrak{m})$ is local. Then, by Lemma 4.7, we can further assume that $(R, \mathfrak{m})$ is complete and hence excellent.

Thus, also by Lemma 4.8, we can assume that R is reduced and that $\overline{R}$ is module-finite over R, where $\overline{R}$ is the integral closure of R in its total fraction ring $(R^\circ)^{-1}R$. Consider $\big(\overline{R} \cap R^{1/q}\big)_{q=1}^{\infty}$, which forms an ascending chain of R-submodules of $\overline{R}$. As $\overline{R}$ is module-finite over R, there exists Q such that $\overline{R} \cap R^{1/q} = \overline{R} \cap R^{1/Q}$ for all $q \geq Q$.

For any $x \in R^\circ$ and any $y \in (xR)^F$, there exists q such that $y^q = ax^q$ for some $a \in R$. This means that $(y/x)^q = a/1$ in $(R^\circ)^{-1}R$, and this implies that $y/x \in \overline{R} \cap R^{1/q}$. By our choice of Q, we get $y/x \in \overline{R} \cap R^{1/Q}$. Thus $(y/x)^Q = b/1$ for some $b \in R$ and hence $y^Q = bx^Q \in (xR)^{[Q]}$. ☐

The next theorem is the main result of this paper. Recall from Theorem 4.2 that a local ring is generalized Cohen–Macaulay if and only if it has a system of parameters that is an unconditioned strong d-sequence.

Theorem 5.2. *Suppose that $(R, \mathfrak{m})$ is a generalized Cohen–Macaulay local ring of prime characteristic p; recall that $\dim(R) = t > 0$. Then there exists Q such that $\big(\big(\sum_{i=1}^{j} x_iR\big)^F\big)^{[Q]} = \big(\sum_{i=1}^{j} x_iR\big)^{[Q]}$ for all subsystems of parameters $x_1, \dots, x_j$ of R.*

Proof. In view of Proposition 5.1, we can assume that $t \geq 2$.

In the first part of the proof, we are going to show that there exists Q_0 such that $\big(\big(\sum_{i=1}^{t} x_iR\big)^F\big)^{[Q_0]} = \big(\sum_{i=1}^{t} x_iR\big)^{[Q_0]}$ for all systems of parameters $\underline{x} = x_1, \dots, x_t$ of R that are unconditioned strong d-sequences.

Let Q_1 be as in Corollary 4.5. Also, by Proposition 4.6, there exists Q_2 such that $\big(\big(\sum_{i=1}^{t} x_iR\big)^F\big)^{[Q_2]} \subseteq \big(\sum_{i=1}^{t} x_i^{Q_2}R\big)^F \bigcap \big(\sum_{i=1}^{t} x_i^{Q_2}R\big)^{\lim}$ for all systems of parameters $\underline{x} = x_1, \dots, x_t$ of R. Set $Q_0 = pQ_1Q_2$. We are going to show that $\big(\big(\sum_{i=1}^{t} x_iR\big)^F\big)^{[Q_0]} = \big(\sum_{i=1}^{t} x_iR\big)^{[Q_0]}$ for all systems of parameters $\underline{x} = x_1, \dots, x_t$ of R that are unconditioned strong d-sequences. Notice that

$$\begin{aligned}\big(\big(\textstyle\sum_{i=1}^{t} x_iR\big)^F\big)^{[pQ_1Q_2]} &= \big(\big(\big(\textstyle\sum_{i=1}^{t} x_iR\big)^F\big)^{[Q_2]}\big)^{[pQ_1]}\\ &\subseteq \big(\big(\textstyle\sum_{i=1}^{t} x_i^{Q_2}R\big)^F \bigcap \big(\textstyle\sum_{i=1}^{t} x_i^{Q_2}R\big)^{\lim}\big)^{[pQ_1]}\end{aligned}$$

by Proposition 4.6. Therefore, it suffices to prove that

$$\big(\big(\textstyle\sum_{i=1}^{t} x_i^{Q_2}R\big)^F \bigcap \big(\textstyle\sum_{i=1}^{t} x_i^{Q_2}R\big)^{\lim}\big)^{[pQ_1]} \subseteq \big(\textstyle\sum_{i=1}^{t} x_i^{Q_2}R\big)^{[pQ_1]}$$

for each system of parameters $\underline{x} = x_1, \dots, x_t$ such that $x_1^{Q_2}, \dots, x_t^{Q_2}$ is an unconditioned strong d-sequence, and so it is enough to prove that

$$\big(\big(\textstyle\sum_{i=1}^{t} x_iR\big)^F \bigcap \big(\textstyle\sum_{i=1}^{t} x_iR\big)^{\lim}\big)^{[pQ_1]} \subseteq \big(\textstyle\sum_{i=1}^{t} x_iR\big)^{[pQ_1]}$$

for all systems of parameters $\underline{x} = x_1, \dots, x_t$ of R that are unconditioned strong d-sequences. We therefore fix a typical such $\underline{x} = x_1, \dots, x_t$. Notice that $\underline{x}$ is also an $\mathfrak{m}$-filter regular sequence in any order, by Remark 3.1(ii).

Let $y \in \big(\sum_{i=1}^{t} x_iR\big)^F \cap \big(\sum_{i=1}^{t} x_iR\big)^{\lim}$. Then there exists $q' = pq$ such that $y^{q'} \in \big(\sum_{i=1}^{t} x_iR\big)^{[q']}$, and without loss of generality we can assume that $q \geq \max\{p, Q_1\}$. We see, from Corollary 3.7(ii), that $y^p \in \big(\big(\sum_{i=1}^{t} x_iR\big)^{\lim}\big)^{[p]} \subseteq \big(\sum_{i=1}^{t} x_i^pR\big)^{\lim} =$

$\sum_{\Lambda \subsetneq [1,t]} x_\Lambda^{p-1}\big(\sum_{i\in\Lambda} x_iR\big)^{(\underline{x})\text{-unm}}$. We can therefore write $y^p = \sum_{\Lambda \subsetneq [1,t]} x_\Lambda^{p-1} h_\Lambda$ with $h_\Lambda \in \big(\sum_{i\in\Lambda} x_iR\big)^{(\underline{x})\text{-unm}}$ for all $\Lambda \subsetneq [1,t]$. Consequently, we have

$$(*_0) \qquad \sum_{\Lambda \subsetneq [1,t]} x_\Lambda^{pq-q} h_\Lambda^q = y^{pq} \in \big(\sum_{i=1}^t x_iR\big)^{[pq]},$$

in which $h_\Lambda^q \in \Big(\big(\sum_{i\in\Lambda} x_iR\big)^{(\underline{x})\text{-unm}}\Big)^{[q]} \subseteq \big(\sum_{i\in\Lambda} x_i^qR\big)^{(\underline{x})\text{-unm}}$ for all $\Lambda \subsetneq [1,t]$ (in view of 3.1(i)).

The immediate goal is to show that $y^{pQ_1} \in \big(\sum_{i=1}^t x_iR\big)^{[pQ_1]}$. To this end, as $y^{pQ_1} = \sum_{\Lambda \subsetneq [1,t]} x_\Lambda^{pQ_1-Q_1} h_\Lambda^{Q_1}$, it is enough for us to prove that

$$(\dagger) \qquad x_\Lambda^{pQ_1-Q_1} h_\Lambda^{Q_1} \in \sum_{i\in\Lambda} x_i^{pQ_1}R \quad \text{for all } \Lambda \text{ such that } \Lambda \subsetneq [1,t].$$

We now prove (†) by induction on $|\Lambda|$, the cardinality of Λ. When $|\Lambda| = 0$, we have $\Lambda = \emptyset$; we recall our conventions that $\sum_{i\in\emptyset} x_i^{pQ_1}R = (0)$ and $x_\emptyset = 1$. When we consider R as a left $R[T,f]$-module as in 1.3, the R-submodule $\Gamma_{\mathfrak{m}}(R)$ is actually a T-torsion $R[T,f]$-submodule. Since $\big(\sum_{i\in\emptyset} x_iR\big)^{(\underline{x})\text{-unm}} = \big(0 : \sum_{i=1}^t x_iR\big)$, we have $h_\emptyset \in \Gamma_{\mathfrak{m}}(R)$, so that $h_\emptyset^{Q_1} = 0$.

Now suppose that $1 \le r < t$, and assume that (†) has been proved for $|\Lambda| < r$. That assumption and $(*_0)$ mean that

$$(*_r) \qquad \sum_{\Lambda \subsetneq [1,t], |\Lambda| \ge r} x_\Lambda^{pq-q} h_\Lambda^q \in \big(\sum_{i=1}^t x_iR\big)^{[pq]}.$$

To prove (†) for $\Lambda \subsetneq [1,t]$ with $|\Lambda| = r$, there is no loss of generality in our assuming that $\Lambda = [1,r]$. For every $\Lambda' \subsetneq [1,t]$ with $|\Lambda'| \ge r$ but $\Lambda' \ne [1,r]$, we have $x_{\Lambda'}^{pq-q} h_{\Lambda'}^q \in \sum_{i=r+1}^t x_i^{pq-q}R$. Therefore, by $(*_r)$, we have

$$x_{[1,r]}^{pq-q} h_{[1,r]}^q \in \sum_{i=1}^t x_i^{pq}R + \sum_{i=r+1}^t x_i^{pq-q}R = \sum_{i=1}^r x_i^{pq}R + \sum_{i=r+1}^t x_i^{pq-q}R.$$

Since $h_{[1,r]}^q \in \big(\sum_{i=1}^r x_i^qR\big)^{(\underline{x})\text{-unm}} \subseteq \big(\big(\sum_{i=1}^r x_i^qR\big) : x_t\big)$, it follows that

$$x_{[1,r]}^{pq-q} h_{[1,r]}^q x_t \in x_{[1,r]}^{pq-q}\big(\sum_{i=1}^r x_i^qR\big) \subseteq \sum_{i=1}^r x_i^{pq}R;$$

this implies that

$$x_{[1,r]}^{pq-q} h_{[1,r]}^q \in \big(\big(\sum_{i=1}^r x_i^{pq}R\big) : x_t\big) = \big(\sum_{i=1}^r x_i^{pq}R\big)^{(\underline{x})\text{-unm}}.$$

Thus $x_{[1,r]}^{pq-q} h_{[1,r]}^q \in \big(\sum_{i=1}^r x_i^{pq}R\big)^{(\underline{x})\text{-unm}} \bigcap \big(\sum_{i=1}^r x_i^{pq}R + \sum_{i=r+1}^t x_i^{pq-q}R\big)$, and this is equal to $\sum_{i=1}^r x_i^{pq}R$ by Theorem 3.8(ii); therefore $h_{[1,r]}^q \in \big(\sum_{i=1}^r x_i^qR\big)^{\lim}$. Therefore $h_{[1,r]}^{Q_1} \in \big(\sum_{i=1}^r x_i^{Q_1}R\big)^{\lim}$ by 4.5, so that $x_{[1,r]}^{pQ_1-Q_1} h_{[1,r]}^{Q_1} \in \sum_{i=1}^r x_i^{pQ_1}R$ by Corollary 3.7(i). This concludes the inductive step in the proof of (†) and so it follows that $y^{pQ_1} \in \big(\sum_{i=1}^t x_iR\big)^{[pQ_1]}$. This is enough to complete the proof that $\Big(\big(\sum_{i=1}^t x_iR\big)^F\Big)^{[Q_0]} = \big(\sum_{i=1}^t x_iR\big)^{[Q_0]}$ for all systems of parameters $\underline{x} = x_1, \ldots, x_t$ of R that are unconditioned strong d-sequences.

Now let h be the integer of 4.2(ii) and let Q_4 be a power of p with $Q_4 \ge h$. Also set $Q = Q_4Q_0$. Let $y_1, \ldots, y_t$ be an arbitrary system of parameters of R. By Theorem 4.2, the system of parameters $y_1^{Q_4}, \ldots, y_t^{Q_4}$ is an unconditioned strong d-sequence.

Therefore, by the first part of the proof,

$$\begin{aligned}\big(\big(\textstyle\sum_{i=1}^t y_iR\big)^F\big)^{[Q_4Q_0]} &= \big(\big(\big(\textstyle\sum_{i=1}^t y_iR\big)^F\big)^{[Q_4]}\big)^{[Q_0]} \subseteq \big(\big(\textstyle\sum_{i=1}^t y_i^{Q_4}R\big)^F\big)^{[Q_0]}\\ &= \big(\textstyle\sum_{i=1}^t y_i^{Q_4}R\big)^{[Q_0]} = \big(\textstyle\sum_{i=1}^t y_iR\big)^{[Q_4Q_0]}.\end{aligned}$$

Thus we have shown that $\big(\big(\sum_{i=1}^t x_iR\big)^F\big)^{[Q]} = \big(\sum_{i=1}^t x_iR\big)^{[Q]}$ for all systems of parameters $x_1,\ldots,x_t$ of R.

Finally, let $j \in \{0,\ldots,t-1\}$. For every $n \in \mathbb{N}_+$, we can apply what we have just proved to the system of parameters $x_1,\ldots x_j, x_{j+1}^n,\ldots,x_t^n$. Thus

$$\begin{aligned}\big(\big(\textstyle\sum_{i=1}^j x_iR\big)^F\big)^{[Q]} &\subseteq \textstyle\bigcap_{n\in\mathbb{N}_+}\big(\big(\sum_{i=1}^j x_iR + \sum_{i=j+1}^t x_i^nR\big)^F\big)^{[Q]}\\ &= \textstyle\bigcap_{n\in\mathbb{N}_+}\big(\sum_{i=1}^j x_iR + \sum_{i=j+1}^t x_i^nR\big)^{[Q]}\\ &= \textstyle\bigcap_{n\in\mathbb{N}_+}\big(\sum_{i=1}^j x_i^QR + \sum_{i=j+1}^t x_i^{nQ}R\big) = \big(\sum_{i=1}^j x_iR\big)^{[Q]}\end{aligned}$$

by Krull's Intersection Theorem. Now the proof is complete. □

Corollary 5.3. *Suppose that $(R,\mathfrak{m})$ is a formally catenary local ring all of whose formal fibres are Cohen–Macaulay. (These hypotheses would be satisfied if R was an excellent local ring.) Assume further that R is equidimensional, of prime characteristic p and of dimension 2. Then there exists Q such that $\big(\big(\sum_{i=1}^l x_iR\big)^F\big)^{[Q]} = \big(\sum_{i=1}^l x_iR\big)^{[Q]}$ for all subsystems of parameters $\underline{x} = x_1,\ldots,x_l$ (where $l \leq 2$, of course) of R.*

Proof. The hypotheses about R are all inherited by $R/\sqrt{0}$, and so, in view of Lemma 4.8, we can assume that R is reduced. But then R is Cohen–Macaulay on the punctured spectrum, and so is a generalized Cohen–Macaulay local ring by 3.4. The result now follows from Theorem 5.2(ii). □

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