[Preliminary Version]

SECOND COEFFICIENTS OF HILBERT-KUNZ FUNCTIONS FOR DOMAINS

MELVIN HOCHSTER AND YONGWEI YAO

Abstract. Let $(R, \mathfrak{m}, k)$ be an excellent (e.g., F-finite) equidimensional local Noetherian ring of prime characteristic p with $\dim(R) = d$, I an ideal of R such that $\lambda(R/I) < \infty$ and M a finitely generated R-module. We study the existence of $\beta(M) \in \mathbb{R}$ such that $\lambda(M/I^{[q]}M) = e_{HK}(I, M)q^d + \beta(M)q^{d-1} + O(q^{d-2})$. We refer to $\beta(M)$ as the second coefficient of the Hilbert-Kunz function. In particular, we show the existence of such $\beta(M)$ when the defining ideal of the singular locus of R has height at least 2.

0. Introduction

Throughout this paper R is a Noetherian commutative ring of prime characteristic p with $\dim(R) = d$ and I is an arbitrarily given ideal of R such that $\lambda_R(R/I) < \infty$. We write $q = p^n$ where n is a varying non-negative integer. For any q, we denote by $I^{[q]}$ the ideal generated by $\{r^q \mid r \in I\}$.

We use $\lambda_R(-)$ (or $\lambda(-)$ if R is understood) to denote the length of an R-module. Given any finitely generated R-module M, there is the Hilbert-Kunz function $e_n(I, M) = \lambda(M/I^{[q]}M)$, which is considered as a map from $\mathbb{N}$ to $\mathbb{N}$. To simplify notation, we often write $e_n(I, M)$ as $e_n(M)$ if no confusion arises.

Remark 0.1. Let R, I, M be as above. It is enough to understand the Hilbert-Kunz functions over local rings: Indeed, let $V(I) = \{\mathfrak{m} \mid \mathfrak{m} \in \operatorname{Spec}(R), I \subseteq \mathfrak{m}\}$, which is a finite set consists of maximal ideals of R. Then we have $e_n(M) = \lambda(M/I^{[q]}M) = \sum_{\mathfrak{m} \in V(I)} \lambda_{R_\mathfrak{m}}(M_\mathfrak{m}/I^{[q]}M_\mathfrak{m}) = \sum_{\mathfrak{m} \in V(I)} e_n(IR_\mathfrak{m}, M_\mathfrak{m})$.

For this reason, we assume R is local most of the time. By the notation $(R, \mathfrak{m}, k)$, we indicate that R is local with its maximal ideal being $\mathfrak{m}$ and its residue field being $k = R/\mathfrak{m}$.

By a result of [Mo], $e_n(I, M) = \alpha(M)q^d + O(q^{d-1})$ for some $\alpha(M) \in \mathbb{R}$. This $\alpha(M)$ is usually called the Hilbert-Kunz multiplicity of M with respect to I and is denoted by $e_{HK}(I, M)$. (Recall that, given functions f, $g : \mathbb{N} \to \mathbb{R}$, we write $f(n) = O(g(n))$ if there exists $C \in \mathbb{R}$ such that $|f(n)| \le |Cg(n)|$ for all $n \in \mathbb{N}$, while we say $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$.)

The above result of [Mo] has been pushed further in [HMM].

Date: October 8, 2009.

2000 *Mathematics Subject Classification.* Primary 13A35; Secondary 13C13, 13H10.

Both authors were partially supported by the National Science Foundation and the second author was also partially supported by the Research Initiation Grant of Georgia State University.

Theorem 0.2 ([HMM]). *Let $(R, \mathfrak{m}, k)$ be an excellent local normal ring of prime characteristic p with a perfect residue field and $\dim(R) = d$. Then $e_n(M) = e_{HK}(I, M)q^d + \beta q^{d-1} + O(q^{d-2})$ for some $\beta \in \mathbb{R}$.*

We are going to study the issue more generally. Let $C_1(R)$ be the quotient of the Grothendieck group $G_0(R)$ by its subgroup spanned by $\{[R/P] \in G_0(R) \mid \dim(R/P) < d-1\}$ (see Notation 1.1 (6)). Our result generalizes [HMM] as follows.

Theorem (Corollary 2.5). *Let $(R, \mathfrak{m}, k)$ be an excellent equidimensional reduced local Noetherian ring of prime characteristic p such that the singular locus of R is defined by an ideal of height at least 2. Then there exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that, for any finitely generated torsion free R-module M, we have*

$$e_n(M) = e_{HK}(I, M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2}).$$

In general, for any finitely generated R-module M (not necessarily torsion free), there exists $b(M) \in \mathbb{R}$ such that

(1) $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$.
(2) $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$.

In proving the above result, we reduce to the F-finite case by the Γ-construction as in [HH]. Recall that R is defined to be F-finite if R is module-finite over $R^q := \{r^q \mid r \in R\}$ for all q (or equivalently, for $q = p$). If R is F-finite, then R is excellent by [Ku]. In particular, its singular locus is a closed subset $V(J) \subseteq \mathrm{Spec}(R)$ defined by an ideal J.

Observe that the above result fails to hold in the following example, in which R is not a domain.

Example 0.3 ([Mo]). Let $R = k[[X, Y]]/(X^5 - Y^5)$ where k is any field of prime characteristic $p \equiv 2$ or $3 \mod 5$. Then $e_n(R) = 5q + c_n$ with $c_n = -4$ when n is even, while $c_n = -6$ when n is odd.

For any R-module M and for any $n \geq 0$, we can derive an R-module structure on the set M by $r \cdot m := r^{p^n} m$ for any $r \in R$ and $m \in M$. We denote the derived R-module by ${}^nM$. In this terminology, we see that R is F-finite if and only if ${}^1R$ (equivalently, ${}^nR$ for every $n \in \mathbb{N}$) is a finitely generated R-module.

Remark 0.4. If $(R, \mathfrak{m}, k)$ is local and $[k : k^p] = p^a$, then it is easy to see that $e_n(I, {}^eM) = \lambda({}^nM/I^{[q]} \cdot {}^eM) = p^{ea}\lambda(M/I^{[qp^e]}M) = p^{ea}e_{n+e}(I, M)$ for any $n, e \in \mathbb{N}$. If we choose e such that $\sqrt{0}^{[p^e]} = 0$, then ${}^eM$ may be considered as a module over $R/\sqrt{0}$. Thus, to study the behavior of $e_n(M)$ when $n \to \infty$, we may assume R is reduced without loss of generality.

1. Sufficient and necessary conditions for the existence of $\beta(M)$

Notation 1.1. Keep the default assumptions on R, I and d.

(1) Denote $\mathrm{Spec}(R, i) = \{P \in \mathrm{Spec}(R) \mid \dim(R/P) = d - i\}$ for any $0 \leq i \leq d$.
(2) Denote $f(M) = \oplus_{P \in \mathrm{Spec}(R,0)}(R/P)^{\lambda_{R_P}(M_P)}$ for any given finitely generated R-module M.

(3) We say that an F-finite ring R satisfies condition $(*)$ if

$$(*) \qquad \lambda_R(\operatorname{Tor}_1^R(R/I^{[q]}, {}^e\!(f(R)))) = O(q^{d-2}) \quad \text{for all sufficiently large } e \in \mathbb{N}.$$

(4) We say that an F-finite local ring $(R, \mathfrak{m}, k)$ satisfies condition $(**)$ if, setting $a = \log_p[k : k^p]$,

$$(**) \qquad \lambda_R(\operatorname{Tor}_1^R(R/I, {}^n\!(f(R)))) = O(q^a q^{d-2}) \quad \text{as } n \to \infty.$$

(5) Denote $W = R \setminus (\cup_{P \in \operatorname{Spec}(R,0)} P)$. We say an R-module M is W-torsion-free if every element of W is a non-zero-divisor on M. Similarly, we say M is W-torsion if $W \cap \operatorname{Ann}_R(M) \neq \emptyset$, which is equivalent to $\dim(M) < d$. Notice that if R is a domain then W-torsion-free (or W-torsion) is the same as torsion-free (or torsion).

(6) Let $G_0(R)$ be the Grothendieck group of R. For any $0 \leq i \leq d$, we denote by $C_i(R)$ the quotient of $G_0(R)$ by the subgroup spanned by $\{[R/P] \in G_0(R) \mid P \in \cup_{j>i} \operatorname{Spec}(R, j)$ i.e., $\dim(R/P) < d - i\}$. Moreover, for any finitely generated R-module M, we denote by $c_i(M)$ the image of $[M]$ in $C_i(R)$. We also denote by $C(R)$ the kernel of the natural map $C_1(R) \to C_0(R)$ and, moreover, we write $c(M) = c_1(M) - c_1(f(M)) \in C(R)$ for any finitely generated R-module M.

(7) Given finitely generated W-torsion R-modules M and N, we write $M \sim N$ if there exists an exact sequence $0 \to K \to M \to N \to C \to 0$ such that $\dim(K \oplus C) \leq d - 2$.

Discussion 1.2. (1). Recall that R is called equidimensional if $\min(R) = \operatorname{Spec}(R, 0)$. If R is catenary (e.g., F-finite) and equidimensional, then $\operatorname{Spec}(R, i)$ consists of all prime ideals P such that $\operatorname{height}(P) = i$.

(2). The natural group homomorphism $G_0(R) \to C_0(R)$, which factors through $C_1(R)$, splits. Hence the natural group homomorphism $C_1(R) \to C_0(R)$ also splits.

(3). Consequently, $C_1(R) \cong C(R) \oplus C_0(R)$. And it is easy to see that, for any finitely generated R-module M, $c(M)$ is exactly the projection of $c_1(M)$ to $C(R)$. For any W-torsion R-module T, we see that $c_1(T) = 0$ if and only if $c(T) = 0$.

(4). If R is normal catenary, then $C(R)$ is the class group of R.

(5). $f(M) = f(N)$ if and only if $c_0(M) = c_0(N)$.

(6). Given finitely generated W-torsion R-module M and N, we see that $M \sim N$ if and only if $M_P \cong N_P$ for all $P \in \operatorname{Spec}(R, 1) \cap (\operatorname{Supp}(M) \cup \operatorname{Supp}(N))$.

(7). Suppose $M \sim N$. Say we have exact sequences $0 \to K \to M \to L \to 0$ and $0 \to L \to N \to C \to 0$ such that $\dim(K \oplus C) \leq d - 2$. From these two exact sequences we see that

$$\left|\big(e_n(M) - \lambda(\operatorname{Tor}_1^R(R/I^{[q]}, M))\big) - \big(e_n(N) - \lambda(\operatorname{Tor}_1^R(R/I^{[q]}, N))\big)\right| \leq O(q^{d-2}) + \lambda(\operatorname{Tor}_2^R(R/I^{[q]}, C)),$$

which relies on the fact that $e_n(T) + \lambda(\operatorname{Tor}_1^R(R/I^{[q]}, T)) = O(q^{\dim(T)})$ for any finitely generated R-module T, which is proved in [HMM, Lemma 1.1]. Assume, moreover, that R satisfies S_2. Then choose an R-regular sequence $\underline{x} = x_1, x_2 \in \operatorname{Ann}(C)$. Since $\operatorname{pd}_R(R/(\underline{x})R) = 2$, we have

$$\lambda(\operatorname{Tor}_2^R(R/I^{[q]}, R/(\underline{x})R)) = \lambda(\operatorname{Tor}_1^R(R/I^{[q]}, R/(\underline{x})R)) - e_n(R/(\underline{x})R),$$

which equal to $O(q^{d-2})$ by [HMM, Lemma 1.1]. Then, as there exists an exact sequence $0 \to D \to (R/(\underline{x})R)^r \to C \to 0$, the long exact sequence forces $\lambda(\mathrm{Tor}_2^R(R/I^{[q]}, C)) = O(q^{d-2})$. Consequently, we have (under the S_2 assumption)

$$e_n(M) - \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = e_n(N) - \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, N)) + O(q^{d-2}).$$

(8). Suppose $M \sim N$ and $(R, \mathfrak{m}, k)$ is local and F-finite with $[k : k^p] = p^a$. Then we also have that

$$e_n(M) - q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^nM)) = e_n(N) - q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^nN)) + O(q^{d-2}),$$

which relies on the fact that $e_n(T) + q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^nT)) + q^{-a}\lambda(\mathrm{Tor}_2^R(R/I, {}^nT)) = O(q^{\dim(T)})$ for any finitely generated R-module T, which is proved in [Se, Page 278, Theorem].

(9). Suppose R is catenary (e.g., F-finite) and equidimensional. For any finitely generated R-module M, we can write $c_1(M) = \sum_{i=1}^t c_1(R/Q_i)$ with $Q_i \in \mathrm{Spec}(R)$. For each Q_i, choose a prime ideal $P_i \subseteq Q_i$ such that $P_i \in \mathrm{Spec}(R, 0)$. Let $K = \oplus_{i=1}^t Q_i/P_i$. Then $c_1(M) + c_1(K) = \sum_{i=1}^t c_1(R/Q_i) + \sum_{i=1}^t (c_1(R/P_i) - c_1(R/Q_i)) = \sum_{i=1}^t c_1(R/P_i) = c_1(f(M)) + c_1(f(K))$, that is $c_1(M \oplus K) = c_1(f(M \oplus K)) \in C_1(R)$. Notice that K is W-torsion-free.

(10). Suppose R is catenary (e.g., F-finite) and equidimensional and $x \in C_1(R)$, say $x = \sum_{i=1}^r c_1(R/Q_i) - \sum_{i=r+1}^s c_1(R/Q_i)$ with $Q_i \in \mathrm{Spec}(R)$. For each Q_i, choose a prime ideal $P_i \subseteq Q_i$ such that $P_i \in \mathrm{Spec}(R, 0)$. Let $M = (\oplus_{i=1}^r R/P_i) \oplus (\oplus_{i=r+1}^s Q_i/P_i))$ and $N = (\oplus_{i=1}^r Q_i/P_i) \oplus (\oplus_{i=r+1}^s R/P_i))$. It is easy to check that $x = c_1(M) - c_1(N)$ and M, N are both W-torsion-free.

Many of the implications in the next Proposition are implicit in [HMM].

Proposition 1.3. *Let $(R, \mathfrak{m}, k)$ be a reduced F-finite equidimensional Noetherian local ring of prime characteristic p with $\dim(R) = d$. Consider the following statements (with $q = p^n$):*

(1) R satisfies $(*)$ and, moreover, for any finitely generated W-torsion R-module T such that $c_1(T) = c_1(f(T)) = 0$ (i.e., $c(T) = c_1(T) = 0$) and all sufficiently large $e \in \mathbb{N}$, $e_n({}^eT) - \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^eT)) = O(q^{d-2})$.

(2) $e_n(M) - e_n(f(M)) = O(q^{d-2})$ for all finitely generated W-torsion-free R-module M such that $c_1(M) = c_1(f(M))$ (i.e., $c(M) = 0$).

(3) $e_n(M) - e_n(N) = O(q^{d-2})$ for all finitely generated W-torsion-free R-modules M and N such that $c_1(M) = c_1(N)$.

(4) There exists a group homomorphism $\tau : C(R) \to \mathbb{R}$ such that $e_n(M) - e_n(N) = \tau(c_1(M) - c_1(N))q^{d-1} + O(q^{d-2})$ for all finitely generated W-torsion-free R-modules M and N satisfying $c_0(M) = c_0(N)$.

(5) There exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that

$$e_n(M) = e_{HK}(I, M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2})$$

for every finitely generated W-torsion-free R-module M.

(6) For any finitely generated W-torsion-free R-module M and for any $e \in \mathbb{N}$, $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^eM)) = O(q^{d-2})$.

(7) For any finitely generated W-torsion-free R-module M, there exists e_0 such that $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^eM)) = O(q^{d-2})$ for all $e_0 \le e \in \mathbb{N}$.

(8) R satisfies $(*)$.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8). If, moreover, R satisfies S_2, then (8) $\Rightarrow$ (1) and, hence, all the above statements are equivalent.

Proof. Denote $a = \log_p[k : k^p]$. The assumption implies that W consists of non-zero-divisors of R.

(1) $\Rightarrow$ (2). There exists an exact sequence $0 \to M \to f(M) \to T \to 0$ so that T is W-torsion and $c_1(T) = 0$. Choose $e \gg 0$ such that $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^e(f(M))) = O(q^{d-2})$ and $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^eT) - e_n({}^eT) = O(q^{d-2})$ by (1). Then there is a long exact sequence

$$\mathrm{Tor}_1^R\left(\frac{R}{I^{[q]}}, {}^e(f(M))\right) \longrightarrow \mathrm{Tor}_1^R\left(\frac{R}{I^{[q]}}, {}^eT\right)$$
$$\longrightarrow \frac{{}^eM}{I^{[q]}\cdot {}^eM} \longrightarrow \frac{{}^e(f(M))}{I^{[q]}\cdot {}^e(f(M))} \longrightarrow \frac{{}^eT}{I^{[q]}\cdot {}^eT} \longrightarrow 0.$$

Thus $p^{ea}e_{n+e}(M) - p^{ea}e_{n+e}(f(M)) = \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^eT)) - e_n({}^eT) - O(q^{d-2}) = O(q^{d-2})$, which implies $e_n(M) - e_n(f(M)) = O(q^{d-2})$.

(2) $\Rightarrow$ (3). By Discussion 1.2(9), there exists a finitely generated W-torsion-free R-module K such that $c_1(M \oplus K) = c_1(f(M \oplus K)) \in C_1(R)$. Notice that $c_1(M) = c_1(N)$ implies that $f(M) = f(N)$ and hence $c_1(N \oplus K) = c_1(f(N \oplus K)) \in C_1(R)$. Now the claim follows from (2) applied to $M \oplus K$ and $N \oplus K$. (3) $\Rightarrow$ (2) is trivial.

(3) $\Rightarrow$ (4). As $c_1({}^1M \oplus N^{p^{d-1+a}}) = c_1({}^1N \oplus M^{p^{d-1+a}})$, we apply (3) to ${}^1M \oplus N^{p^{d-1+a}}$ and ${}^1N \oplus M^{p^{d-1+a}}$, which gives that

$$\begin{aligned} e_n({}^1M \oplus N^{p^{d-1+a}}) - e_n({}^1N \oplus M^{p^{d-1+a}}) &= O(q^{d-2}) && \text{that is} \\ (e_n({}^1M) - e_n({}^1N)) - p^{d-1+a}(e_n(M) - e_n(N)) &= O(q^{d-2}) && \text{that is} \\ (e_{n+1}(M) - e_{n+1}(N)) - p^{d-1}(e_n(M) - e_n(N)) &= O(q^{d-2}) && \text{which gives} \\ e_n(M) - e_n(N) &= t(M,N)q^{d-1} + O(q^{d-2}) \end{aligned}$$

for some $t(M,N) \in \mathbb{R}$, in which t is viewed as a map. For every element $x \in C(R)$, we define $\tau(x) = t(M,N)$ provided $x = c_1(M) - c_1(N)$ with M and N W-torsion-free finitely generated over R (cf. Discussion 1.2(10)). To check well-definedness, say $x = c_1(M') - c_1(N')$ with M' and N' W-torsion-free. Then $c_1(M \oplus N') = c_1(M' \oplus N)$, which implies $e_n(M \oplus N') = e_n(M' \oplus N) + O(q^{d-2})$, that is, $e_n(M) - e_n(N) = e_n(M') - e_n(N') + O(q^{d-2})$ by (4), which forces $t(M,N) = t(M',N')$. Now that we have showed that $\tau : C(R) \to \mathbb{R}$ is well-defined, it is straightforward to verify that τ is a group homomorphism.

(4) $\Rightarrow$ (5). As $c_0({}^1M) = c_0(M^{p^{d+a}})$, we apply (4) to ${}^1M$ and $M^{p^{d+a}}$, which gives that (with $\tau(c_1({}^1M) - c_1(M^{p^{d+a}})) = b'(M) = p^a b''(M) \in \mathbb{R}$)

$$\begin{aligned} e_n({}^1M) - e_n(M^{p^{d+a}}) &= b'(M)q^{d-1} + O(q^{d-2}) && \text{that is} \\ e_n({}^1M) - p^{d+a}e_n(M) &= b'(M)q^{d-1} + O(q^{d-2}) && \text{that is} \\ e_{n+1}(M) - p^d e_n(M) &= b''(M)q^{d-1} + O(q^{d-2}) && \text{which gives} \\ e_n(M) &= e_{HK}(I,M)q^d + b(M)q^{d-1} + O(q^{d-2}) && \text{(cf. [HMM, Theorem 1.11])} \end{aligned}$$

with $b(M) = b''(M)/(p^{d-1} - p^d) = \tau(c_1({}^1M) - c_1(M^{p^{d+a}}))/(p^{d-1+a} - p^{d+a})$, in which b is considered as a map. For every element $x \in C_1(R)$, set $\beta(x) = b(M) - b(N)$ if $x = c_1(M) - c_1(N)$ with M and N finitely generated W-torsion-free R-modules (cf. Discussion 1.2(9)). It is straightforward to check that $\beta : C_1(R) \to \mathbb{R}$ is a well-defined group homomorphism.

(5) $\Rightarrow$ (3). This is trivial as $c_1(M) \mapsto e_{HK}(I, M)$ is well-defined and determines a group homomorphism from $C_1(R)$ to $\mathbb{R}$.

(5) $\Rightarrow$ (6). It suffice to prove $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = O(q^{d-2})$ as the assumption of M being W-torsion-free implies ${}^eM$ being W-torsion-free for all $e \in \mathbb{N}$. Choose an exact sequence $0 \to M' \to G \to M \to 0$ such that G is free of finite rank over R. Then G and hence M' are W-torsion-free. Now $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = e_n(M') - e_n(G) + e_n(M) = \big(e_{HK}(I, M') - e_{HK}(I, G) + e_{HK}(I, M)\big)q^d + \big(\beta(c_1(M')) - \beta(c_1(G)) + \beta(c_1(M))\big)q^{d-1} + O(q^{d-2}) = O(q^{d-2})$.

(6) $\Rightarrow$ (7). This is obvious.

(7) $\Rightarrow$ (8). This follows immediately as R is W-torsion-free.

(8) $\Rightarrow$ (1) in case R satisfies S_2. Let A be the free abelian group generated by the set of all isomorphic classes $\{[R/Q] \mid Q \in \mathrm{Spec}(R, 1)\}$. Then $C(R)$ is a quotient of A modulo a subgroup generated by $\{\sum_{Q \in \mathrm{Spec}(R,1)} \lambda_{R_Q}((R/(P + xR))_Q)[R/Q] \mid P \in \mathrm{Spec}(R, 0), x \in R \setminus P\}$.

The assumption $c_1(T) = c(T) = 0$ implies that there exist $r \le s$, $P_i \in \mathrm{Spec}(R, 0)$, $x_i \notin P_i$ for $1 \le i \le s$ such that

$$\sum_{Q \in \mathrm{Spec}(R,1)} \lambda_{R_Q}(T_Q)[R/Q] + \sum_{i=1}^{r} \sum_{Q \in \mathrm{Spec}(R,1)} \lambda_{R_Q}((R/(P_i + x_iR))_Q)[R/Q]$$
$$= \sum_{i=r+1}^{s} \sum_{Q \in \mathrm{Spec}(R,1)} \lambda_{R_Q}((R/(P_i + x_iR))_Q)[R/Q]$$

as elements in the (free abelian) group A.

Choose e_0 such that the statement of $(*)$ always holds for $e \ge e_0$ and such that

$$\sqrt{\mathrm{Ann}_R(T \oplus (\oplus_{i=1}^s R/(P_i + x_iR)))}^{[p^{e_0}]} \subseteq \mathrm{Ann}_R(T \oplus (\oplus_{i=1}^s R/(P_i + x_iR))).$$

Then for all $e \ge e_0$, we have ${}^eT \oplus (\oplus_{i=1}^r {}^e(R/(P_i + x_iR)) \sim \oplus_{i=r+1}^s {}^e(R/(P_i + x_iR))$. Therefore, to prove the claim of (1), it suffices to prove that, for any $P \in \mathrm{Spec}(R, 0), x \notin P, e_0 \le e \in \mathbb{N}$, we always have

$$e_n({}^e(R/(P + xR))) - \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^e(R/(P + xR)))) = O(q^{d-2}).$$

Indeed, there is an exact sequence $0 \to {}^e(R/P) \to {}^e(R/P) \to {}^e(R/(P + xR)) \to 0$, which gives a long exact sequence

$$\mathrm{Tor}_1^R\Big(\frac{R}{I^{[q]}}, {}^e(R/P)\Big) \longrightarrow \mathrm{Tor}_1^R\Big(\frac{R}{I^{[q]}}, {}^e(R/(P_i + x_iR))\Big)$$
$$\longrightarrow \frac{{}^e(R/P)}{I^{[q]} \cdot {}^e(R/P)} \longrightarrow \frac{{}^e(R/P)}{I^{[q]} \cdot {}^e(R/P)} \longrightarrow \frac{{}^e(R/(P_i + x_iR))}{I^{[q]} \cdot {}^e(R/(P_i + x_iR))} \longrightarrow 0,$$

which implies $e_n({}^{e}(R/(P+xR))) - \lambda(\mathrm{Tor}_1^R(R/I^{[q]}, {}^{e}(R/(P+xR)))) = e_n({}^{e}(R/P)) - e_n({}^{e}(R/P)) + O(q^{d-2}) = O(q^{d-2})$.

Now the proof is complete. □

Example 1.4. Suppose $(R, \mathfrak{m}, k)$ is normal. Then statement (2) of Proposition 1.3 is verified in [HMM, Theorem 1.4]. Therefore statements (1) through (8) of Proposition 1.3 all hold.

Proposition 1.5. *Let $(R, \mathfrak{m}, k)$ be a reduced F-finite equidimensional local Noetherian ring of prime characteristic p. Denote $[k : k^p] = p^a$. Consider the following statements (with $q = p^n$):*

(1) R satisfies $(**)$.

(2) R satisfies $(**)$ and, moreover, for any finitely generated W-torsion R-module T such that $c(T) = 0$, $e_n(T) - q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^{n}T)) = O(q^{d-2})$.

(3) $e_n(M) - e_n(f(M)) = O(q^{d-2})$ for all finitely generated W-torsion-free R-module M such that $c_1(M) = c_1(f(M))$ (i.e., $c(M) = 0$).

(4) $e_n(M) - e_n(N) = O(q^{d-2})$ for all finitely generated W-torsion-free R-modules M and N such that $c_1(M) = c_1(N)$.

(5) There exists a group homomorphism $\tau : C(R) \to \mathbb{R}$ such that $e_n(M) - e_n(N) = \tau(c_1(M) - c_1(N))q^{d-1} + O(q^{d-2})$ for all finitely generated W-torsion-free R-modules M and N satisfying $c_0(M) = c_0(N)$.

(6) There exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that

$$e_n(M) = e_{HK}(I, M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2})$$

for every finitely generated W-torsion-free R-module M.

(7) $q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^{n}M)) = O(q^{d-2})$ for any finitely generated W-torsion-free R-module M.

Then (7) ⇔ (1) ⇒ (2) ⇒ (3) ⇔ (4) ⇔ (5) ⇔ (6).

Proof. The proof is very similar to (and actually simpler than) the proof of Proposition 1.3.

(1) ⇒ (2). Let A be the free abelian group generated by the set of all isomorphic classes $\{[R/Q] \mid Q \in \mathrm{Spec}(R, 1)\}$. Then $C(R)$ is a quotient of A modulo a subgroup generated by $\{\sum_{Q \in \mathrm{Spec}(R,1)} \lambda_{R_Q}((R/(P + xR))_Q)[R/Q] \mid P \in \mathrm{Spec}(R, 0), x \in R \setminus P\}$.

The assumption $c_1(T) = c(T) = 0$ implies that there exist $r \le s$, $P_i \in \mathrm{Spec}(R, 0)$, $x_i \notin P_i$ for $1 \le i \le s$ such that

$$\sum_{Q \in \mathrm{Spec}(R,1)} \lambda_{R_Q}(T_Q)[R/Q] + \sum_{i=1}^{r} \sum_{Q \in \mathrm{Spec}(R,1)} \lambda_{R_Q}((R/(P_i + x_iR))_Q)[R/Q]$$
$$= \sum_{i=r+1}^{s} \sum_{Q \in \mathrm{Spec}(R,1)} \lambda_{R_Q}((R/(P_i + x_iR))_Q)[R/Q]$$

as elements in the (free abelian) group A.

Choose n_0 such that

$$\sqrt{\mathrm{Ann}_R(T \oplus (\oplus_{i=1}^{s} R/(P_i + x_iR)))}^{[p^{n_0}]} \subseteq \mathrm{Ann}_R(T \oplus (\oplus_{i=1}^{s} R/(P_i + x_iR))).$$

Then for all $n \geq n_0$, we have ${}^{n}T \oplus (\oplus_{i=1}^{r} {}^{n}(R/(P_i + x_iR)) \sim \oplus_{i=r+1}^{s} {}^{n}(R/(P_i + x_iR))$. Therefore, to prove the claim of (2), it suffices to prove that, for any $P \in \operatorname{Spec}(R, 0)$ and $x \notin P$, we always have

$$\lambda(R/I \otimes {}^{n}(R/(P + xR))) - \lambda(\operatorname{Tor}_1^R(R/I, {}^{n}(R/(P + xR)))) = O(q^{d-2}q^a).$$

Indeed, there is an exact sequence $0 \to {}^{n}(R/P) \to {}^{n}(R/P) \to {}^{n}(R/(P + xR)) \to 0$, which gives a long exact sequence

$$\begin{aligned}\operatorname{Tor}_1^R\Big(\frac{R}{I}, {}^{n}(R/P)\Big) \longrightarrow \operatorname{Tor}_1^R\Big(\frac{R}{I}, {}^{n}(R/(P_i + x_iR))\Big)&\\ \longrightarrow \frac{{}^{n}(R/P)}{I \cdot {}^{n}(R/P)} \longrightarrow \frac{{}^{n}(R/P)}{I \cdot {}^{n}(R/P)} \longrightarrow \frac{{}^{n}(R/(P_i + x_iR))}{I \cdot {}^{n}(R/(P_i + x_iR))} \longrightarrow 0&,\end{aligned}$$

which implies

$$\begin{aligned}&\lambda(R/I \otimes {}^{n}(R/(P + xR))) - \lambda(\operatorname{Tor}_1^R(R/I, {}^{n}(R/(P + xR))))\\ &\qquad = \lambda(R/I \otimes {}^{n}(R/P)) - \lambda(R/I \otimes {}^{n}(R/P)) + O(q^{d-2}q^a) = O(q^{d-2}q^a).\end{aligned}$$

(2) $\Rightarrow$ (3). There exists an exact sequence $0 \to M \to f(M) \to T \to 0$ so that T is W-torsion and $c_1(T) = 0$. Then, as $n \to \infty$, $\lambda(\operatorname{Tor}_1^R(R/I, {}^{n}(f(M))) = O(q^{d-2}q^a)$ and $\lambda(\operatorname{Tor}_1^R(R/I, {}^{n}T) - \lambda(R/I \otimes {}^{n}T) = O(q^{d-2}q^a)$ by (1). Also there is a long exact sequence

$$\begin{aligned}\operatorname{Tor}_1^R\Big(\frac{R}{I}, {}^{n}(f(M))\Big) \longrightarrow \operatorname{Tor}_1^R\Big(\frac{R}{I}, {}^{n}T\Big)&\\ \longrightarrow \frac{{}^{n}M}{I \cdot {}^{n}M} \longrightarrow \frac{{}^{n}(f(M))}{I \cdot {}^{n}(f(M))} \longrightarrow \frac{{}^{n}T}{I \cdot {}^{n}T} \longrightarrow 0&.\end{aligned}$$

Thus $q^a e_n(M) - q^a e_n(f(M)) = \lambda(\operatorname{Tor}_1^R(R/I, {}^{n}T)) - q^a e_n(T) - O(q^{d-2}q^a) = O(q^{d-2}q^a)$, which implies $e_n(M) - e_n(f(M)) = O(q^{d-2})$.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). This is proved in Proposition 1.3.

(7) $\Rightarrow$ (1). This follows immediately as R is W-torsion-free.

(1) $\Rightarrow$ (7). By Discussion 1.2(8), there exists a finitely generated W-torsion-free R-module K such that $c_1(M \oplus K) = c_1(f(M \oplus K)) \in C_1(R)$. Thus, as it suffices to prove the claim for $M \oplus K$, we may assume $c_1(M) = c_1(f(M))$ without loss of generality. There exists an exact sequence $0 \to f(M) \to M \to T \to 0$ so that $c_1(T) = 0$ and T is W-torsion. Then, for any $n \in \mathbb{N}$, there is a long exact sequence

$$\begin{aligned}\operatorname{Tor}_1^R\Big(\frac{R}{I}, {}^{n}(f(M))\Big) \longrightarrow \operatorname{Tor}_1^R\Big(\frac{R}{I}, {}^{n}M\Big) \longrightarrow \operatorname{Tor}_1^R\Big(\frac{R}{I}, {}^{n}T\Big)&\\ \longrightarrow \frac{{}^{n}(f(M))}{I \cdot {}^{n}(f(M))} \longrightarrow \frac{{}^{n}M}{I \cdot {}^{n}M} \longrightarrow \frac{{}^{n}T}{I \cdot {}^{n}T} \longrightarrow 0&,\end{aligned}$$

which gives the desired conclusion

$$\begin{aligned}&\lambda(\operatorname{Tor}_1^R(R/I, {}^{n}M))\\ &= q^a\big(e_n(M) - e_n((f(M))\big) + \big(q^a e_n(T) - \lambda(\operatorname{Tor}_1^R(R/I, {}^{n}T))\big) - O(q^a q^{d-2})\\ &= q^a O(q^{d-2}) + q^a O(q^{d-2}) - O(q^a q^{d-2})\\ &= O(q^a q^{d-2}),\end{aligned}$$

by $(**)$ applied to $f(M)$, (3) applied to M, and by (2) applied to T. □

2. Applications

Theorem 2.1 (See [HMM, Theorem 1.12])**.** *Let $(R, \mathfrak{m}, k)$ be an F-finite reduced equidimensional Noetherian local ring of prime characteristic p satisfying condition* (5) *of Proposition 1.3 or condition* $(**)$*. Then there exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ and, for any finitely generated R-module M, there exists $b(M) \in \mathbb{R}$ such that*

(1) $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$.

(2) $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$*; and* $q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^{n}M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$ *in case of* $(**)$*.*

Proof. As condition $(**)$ implies Proposition 1.5(6), which is the same as Proposition 1.3(5), we may simply assume Proposition 1.3(5).

Let $T = \{x \in M \,|\, x/1 = 0 \in W^{-1}M\}$ be the W-torsion submodule of M. Then $M' = M/T$ is W-torsion-free and there is an exact sequence $0 \to T \to M \to M' \to 0$. Observe that $e_{HK}(I, M) = e_{HK}(I, M')$. There also exists an exact sequence $0 \to N \to G \to M \to 0$ such that G is free of finite rank over R. Then G and hence M' are W-torsion-free.

Let $\beta : C_1(R) \to \mathbb{R}$ be as in Proposition 1.3(5). Then apply $R/I^{[q]} \otimes_R$ to $0 \to T \to M \to M' \to 0$ and the same argument as in the proof of [HMM, Theorem 1.12] shows part (1), that is $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$ for some $b(M) \in \mathbb{R}$.

To prove (2), notice that the long exact sequence of Tor gives $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = e_n(N) - e_n(G) + e_n(M) = \big(e_{HK}(I, N) - e_{HK}(I, G) + e_{HK}(I, M)\big)q^d + \big(\beta(c_1(N)) - \beta(c_1(G)) + b(M)\big)q^{d-1} + O(q^{d-2}) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$. In case of $(**)$, notice that the long exact sequence of Tor also gives $\lambda(\mathrm{Tor}_1^R(R/I, {}^{n}M)) = q^a e_n(N) - q^a e_n(G) + q^a e_n(M) + O(q^a q^{d-2}) = \big(e_{HK}(I, N) - e_{HK}(I, G) + e_{HK}(I, M)\big)q^a q^d + \big(\beta(c_1(N)) - \beta(c_1(G)) + b(M)\big)q^a q^{d-1} + O(q^a q^{d-2}) = \big(b(M) - \beta(c_1(M))\big)q^a q^{d-1} + O(q^a q^{d-2})$, that is, $q^{-a}\lambda(\mathrm{Tor}_1^R(R/I, {}^{n}M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$. □

Corollary 2.2. *Let $(R, \mathfrak{m}, k)$ be an F-finite equidimensional Noetherian local ring of prime characteristic p such that $R/\sqrt{0}$ satisfies condition* (5) *of Proposition 1.3 or condition* $(**)$*. Then, for any finitely generated R-module M, there exists $b(M) \in \mathbb{R}$ such that $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$.*

Proof. There exists e such that $\sqrt{0}^{[p^e]} = 0$. Then ${}^{e}M$ may be considered as a finitely generated module over $R/\sqrt{0}$. As it suffices to prove the claim for ${}^{e}M$, we assume R is reduced and satisfies condition (5) of Proposition 1.3 or condition $(**)$ without loss of generality. Now the claim follow immediately from Theorem 2.1. (See Remark 0.4.) □

Theorem 2.3. *Let $(R, \mathfrak{m}, k)$ be an F-finite Noetherian local equidimensional reduced ring of prime characteristic p. Suppose there is an module-finite extension ring R' of R in the total fraction ring of R such that (a) $R'_{\mathfrak{n}}$ satisfies condition* (2) *of Proposition 1.3 or condition* $(**)$ *for every $\mathfrak{n} \in V(IR') \subseteq \mathrm{Spec}(R')$, and (b) $\mathrm{Ann}_R(R'/R)$ has height at least 2. Then there exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that, for any finitely generated torsion free R-module M, we have*

$$e_n(M) = e_{HK}(I, M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2}).$$

In general, for any finitely generated R-module M (not necessarily torsion free), there exists $b(M) \in \mathbb{R}$ such that

(1) $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$.

(2) $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$.

Proof. As condition $(**)$ implies Proposition 1.5(3), which is the same as Proposition 1.3(2), we may simply assume Proposition 1.3(2)(3).

Throughout this proof, we will denote $M \otimes_R R'$ by M' and denote the torsion submodule of M' by $T(M')$ for any given R-module M. Thus $M'/T(M')$ is a torsion-free R'-module. As an R'-module, $e_n(IR', M') = \lambda_{R'}(M'/I^{[p^n]}M')$. As an R-module, $e_n(I, M') = \lambda_R(M'/I^{[p^n]}M')$.

For any exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of finitely generated R-modules, there is an induced exact sequence $0 \to K \to M_1' \to M' \to M_2' \to 0$ for some finitely generated R'-module K. As $(R')_P = R_P$ (and hence $K_P = 0$) for any $P \in \mathrm{Spec}(R, 0) \cup \mathrm{Spec}(R, 1)$, we see that $\dim_{R'}(K) = \dim_R(K) < d - 1$. This implies that $c_1(M) \mapsto c_1(M \otimes_R R')$ defines a group homomorphism $C_1(R) \to C_1(R')$.

For any finitely generated torsion-free R-module M, we have an induced long exact sequence $\mathrm{Tor}_1^R(M, R'/R) \to M \to M' \to M \otimes_R R'/R \to 0$, which actually implies an exact sequence $0 \to M \to M' \to M \otimes_R R'/R \to 0$ since M is torsion-free while $\mathrm{Tor}_1^R(M, R'/R)$ is torsion. This implies that $e_n(I, M) - e_n(I, M') = O(q^{d-2})$ by [HMM, Lemma 1.1]. Moreover, for any $P \in \mathrm{Spec}(R, 0) \cup \mathrm{Spec}(R, 1)$, we see that $(M')_P \cong M_P$ is torsion-free, meaning $(T(M'))_P = 0$. Hence $\dim_{R'}(T(M')) = \dim_R(T(M')) < d - 1$. Also, notice that, any $\mathfrak{n} \in V(IR')$, $\dim(R'_{\mathfrak{n}}) = \dim(R)$ by the dimension formula. Consequently, $c_1(M'_{\mathfrak{n}}) = c_1(M'_{\mathfrak{n}}/T(M')_{\mathfrak{n}}) \in C_1(R'_{\mathfrak{n}})$ and $e_n(IR'_{\mathfrak{n}}, M'_{\mathfrak{n}}) = e_n(IR'_{\mathfrak{n}}, M'_{\mathfrak{n}}/T(M')_{\mathfrak{n}}) + O(q^{d-2})$ for every $\mathfrak{n} \in V(IR')$. It is easy to see that $M'_{\mathfrak{n}}/T(M')_{\mathfrak{n}}$ is a torsion-free module over $R'_{\mathfrak{n}}$.

By Proposition 1.3 and Theorem 2.1, it suffices to show that $e_n(I, M) - e_n(I, N) = O(q^{d-2})$ for all finitely generated torsion-free R-modules M and N provided that $c_1(M) = c_1(N)$. For any such M and N, we have $c_1(M') = c_1(N') \in C_1(R')$ and hence, by the paragraph above, $c_1(M'_{\mathfrak{n}}/T(M')_{\mathfrak{n}}) = c_1(N'_{\mathfrak{n}}/T(N')_{\mathfrak{n}}) \in C_1(R'_{\mathfrak{n}})$ for every $\mathfrak{n} \in V(IR')$. By the assumption on $R'_{\mathfrak{n}}$, we have $e_n(IR'_{\mathfrak{n}}, M'_{\mathfrak{n}}/T(M')_{\mathfrak{n}}) = e_n(IR'_{\mathfrak{n}}, N'_{\mathfrak{n}}/T(N')_{\mathfrak{n}}) + O(q^{d-2})$ for every $\mathfrak{n} \in V(IR')$, which implies $e_n(IR'_{\mathfrak{n}}, M'_{\mathfrak{n}}) = e_n(IR'_{\mathfrak{n}}, N'_{\mathfrak{n}}) + O(q^{d-2})$ for every $\mathfrak{n} \in V(IR')$ by last paragraph. By Remark 0.1, we get $e_n(IR', M') = e_n(IR', N') + O(q^{d-2})$, which implies the desired result that $e_n(I, M) = e_n(I, N) + O(q^{d-2})$ from what have been shown in the last paragraph. $\square$

As a corollary, we conclude that it suffices to consider the S_2 rings as far as the current issue is concerned. Recall that the S_2-ification of an F-finite local Noetherian reduced ring always exists.

Corollary 2.4. *Let $(R, \mathfrak{m}, k)$ be an F-finite equidimensional local Noetherian reduced ring of prime characteristic p and R' be the S_2-ification of R. Suppose R' satisfies condition $(*)$ or $(**)$ locally at every $\mathfrak{n} \in V(IR')$. Then there exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that, for any finitely generated torsion free R-module M, we have*

$$e_n(M) = e_{HK}(I, M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2}).$$

In general, for any finitely generated R-module M (not necessarily torsion free), there exists $b(M) \in \mathbb{R}$ such that

(1) $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$.
(2) $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$.

Proof. Since R' has S_2, Proposition 1.3(2) is satisfied over R'. By construction of R', $\mathrm{Ann}_R(R'/R)$, as an ideal of R, has height at least 2. Now apply Theorem 2.3. □

A special case of the above corollary is the following.

Corollary 2.5. *Let $(R, \mathfrak{m}, k)$ be an excellent equidimensional Noetherian reduced ring of prime characteristic p such that the singular locus of R is defined by an ideal of height at least 2. Then there exists a group homomorphism $\beta : C_1(R) \to \mathbb{R}$ such that, for any finitely generated torsion free R-module M, we have*

$$e_n(M) = e_{HK}(I, M)q^d + \beta(c_1(M))q^{d-1} + O(q^{d-2}).$$

In general, for any finitely generated R-module M (not necessarily torsion free), there exists $b(M) \in \mathbb{R}$ such that
(1) $e_n(M) = e_{HK}(I, M)q^d + b(M)q^{d-1} + O(q^{d-2})$.
(2) $\lambda(\mathrm{Tor}_1^R(R/I^{[q]}, M)) = \big(b(M) - \beta(c_1(M))\big)q^{d-1} + O(q^{d-2})$.

Proof. By the Γ-construction, we may assume R is F-finite without loss of generality. (First, notice that $\widehat{R}$ remains equidimensional and reduced with its singular locus defined by an ideal of height at least 2. Then, by the Γ-construction (see [HH, Section 6]), there exists a faithfully flat local and purely inseparable extension $(\widehat{R}^\Gamma, \mathfrak{m}\widehat{R}^\Gamma)$ of $(\widehat{R}, \mathfrak{m}\widehat{R})$ such that $\widehat{R}^\Gamma$ is an F-finite, reduced and equidimensional local ring. Moreover, by choosing Γ small enough, one can make sure that $\widehat{R}$ and $\widehat{R}^\Gamma$ have the same singular locus under the natural homeomorphism $\mathrm{Spec}(\widehat{R}) \cong \mathrm{Spec}(\widehat{R}^\Gamma)$. Thus, the singular locus of $\widehat{R}^\Gamma$ is defined by an ideal of height at least 2. It is easy to see that there is a well-defined group homomorphism $C_1(R) \to C_1(\widehat{R}^\Gamma)$ induced by $[M] \mapsto [M \otimes_R \widehat{R}^\Gamma]$. Moreover, as $\mathfrak{m}\widehat{R}^\Gamma$ is the maximal ideal of $\widehat{R}^\Gamma$, the Hilbert-Kunz functions $e_n(I, M)$ over R and $e_n(I\widehat{R}^\Gamma, M \otimes_R \widehat{R}^\Gamma)$ over $\widehat{R}^\Gamma$ are the same for any finitely generated R-module M.)

Let R' be the integral closure of R in its total fraction ring. Then $\mathrm{Ann}_R(R'/R)$ is an ideal of R with height at least 2. (Therefore R' is the S_2-ification of R.) By [HMM], R' satisfies Proposition 1.3(2). Now apply Theorem 2.3 or Corollary 2.4. □

Remark 2.6. Let R' be as in the above proof and let $\mathfrak{A} := (R :_R R') = \mathrm{Ann}_R(R'/R)$. Then $\mathfrak{A}M$ is an R-submodule of M and $\dim(M/\mathfrak{A}M) \leq \dim(R) - 2$ since $\dim(R/\mathfrak{A}) \leq \dim(R) - 2$. But, as $\mathfrak{A}$ is also an ideal of R', $\mathfrak{A}M$ is an R'-module and the result of [HMM] applies. This should give an alternate proof to Corollary 2.5.

Example 2.7. Let $S = k[X_1, X_2, \ldots, X_d]$ where k is a field of characteristic p and $d \geq 2$, and $k \subseteq R \subseteq S$ such that $X_1^{n_1}X_2^{n_2}\cdots X_d^{n_d} \in R$ for all $n_1 + n_2 + \cdots + n_d \gg 0$. Then $\mathrm{height}_R(S/R) = d$ and the above result applies. Notice that R is not normal unless $R = S$.

Similarly, let $S = k[[X_1, X_2, \ldots, X_d]]$ where k is a field of characteristic p and $d \geq 2$, and $k \subseteq R \subseteq S$ such that $X_1^{n_1}X_2^{n_2}\cdots X_d^{n_d}S \subset R$ for all $n_1 + n_2 + \cdots + n_d \gg 0$. Then $\mathrm{height}_R(S/R) = d$ and the above result applies. Notice that R is not normal unless $R = S$.

References

[HH] M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change,* Tans. Amer. Math. Soc. **346** (1994), 1–62. MR **95d:**13007

[HMM] C. Huneke, M. McDermott and P. Monsky, *Hilbert-Kunz function for normal rings,* Math. Res. Lett., **11** (2004), no. 4, 539–546. MR 2092906

[Ku] E. Kunz, *On Noetherian rings of characteristic p,* Amer. Jour. of Math. **98** (1976), no 4, 999–1013. MR 55 #5612

[Mo] P. Monsky, *The Hilbert-Kunz function,* Math. Ann. **263** (1983), no. 1, 43–49. MR **84k:**13012

[Se] G. Seibert, *Complexes with homology of finite length and Frobenius functors,* J. Algebra **125** (1989), no. 2, 278–287. MR **90j:**13012

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109
E-mail address: hochster@umich.edu

Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303
E-mail address: yyao@gsu.edu