F-RATIONAL SIGNATURE AND DROPS IN THE HILBERT-KUNZ MULTIPLICITY

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p. We define the F-rational signature of R, denoted by $\mathrm{r}(R)$, as the infimum, taken over pairs of ideals $I \subsetneq J$ such that I is generated by a system of parameters and J is a strictly larger ideal, of the drops $\mathrm{e}_{\mathrm{HK}}(I, R)) - \mathrm{e}_{\mathrm{HK}}(J, R)$ in the Hilbert-Kunz multiplicity. If R is excellent, then R is F-rational if and only if $\mathrm{r}(R) > 0$. The proof of this fact depends on the following result in the sequel: Given an $\mathfrak{m}$-primary ideal I in R, there exists a positive $\delta_I \in \mathbb{R}^+$ such that, for any ideal $\mathfrak{m} \supseteq J \supsetneq I$, $\mathrm{e}_{\mathrm{HK}}(I, R) - \mathrm{e}_{\mathrm{HK}}(J, R)$ is either 0 or $\geqslant \delta_I$. We study how the F-rational signature behaves under deformation, flat local ring extension, and localization.

0. Introduction

Throughout this paper rings are assumed to be Noetherian rings of prime characteristic p (unless stated otherwise explicitly). By $(R, \mathfrak{m}, k)$, we mean that R is a local ring, with $\mathfrak{m}$ the unique maximal ideal and $k = R/\mathfrak{m}$ the residue field of R.

Hochster and Huneke introduced and developed tight closure theory for rings of prime characteristic p; see [HH1] or Definition 1.1. One can define several types of rings (singularities) via tight closure, including F-rational rings, weakly F-regular rings, F-regular rings, and stronglyly F-regular rings; see [HH1] and [HH2] for detailed definitions. For example, we say that a ring R is F-rational if all parameter ideals are tightly closed (cf. Definition 1.2). It has been established that F-rationality in prime characteristic corresponds to rational singularity in characteristic 0 via reduction to characteristic p; see [Sm2], [Ha] and also [MS].

There are several invariants defined for (local) rings of prime characteristic p. One such invariant is the F-signature of R, denoted $\mathrm{s}(R)$; see Definition 1.12 and Definition 1.13. It has been shown in [HL] and [AL] that an excellent local ring R is strongly F-regular if and only if $\mathrm{s}(R) > 0$; see Theorem 1.14 for more details.

Another numerical invariant defined for $(R, \mathfrak{m}, k)$ is the Hilbert-Kunz multiplicity $\mathrm{e}_{\mathrm{HK}}(I, R)$, in which I is an $\mathfrak{m}$-primary ideal of R; cf. Notation 1.10. This invariant is closely connected to tight closure in the sense that (under mild conditions) $\mathfrak{m}$-parimary ideals I and J have the same tight closure if and only if $\mathrm{e}_{\mathrm{HK}}(I, R) = \mathrm{e}_{\mathrm{HK}}(J, R)$; see Theorem 1.11 for details.

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The focus of this paper is to define a new invariant, the *F-rational signature*.

Definition 0.1. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p. We define the F-rational signature of R, denoted $\mathrm{r}_R(R)$ or simply $\mathrm{r}(R)$, as follows (here s.o.p. stands for system of parameters of R, and we agree that $\mathrm{e}_{\mathrm{HK}}(J, R) = 0$ if $J = R$)

$$\mathrm{r}_R(R) = \inf\{\mathrm{e}_{\mathrm{HK}}(I, R) - \mathrm{e}_{\mathrm{HK}}(J, R) \,|\, I \text{ is generated by a s.o.p. and } I \subsetneq J\}.$$

In the above definition, a priori, I runs over all ideals of R generated by a system of parameters. In Section 2, we show that, under mild conditions, $\mathrm{r}(R)$ can be defined via one single choice of ideal generated by a system of parameters; see Theorem 2.5.

In Section 3, we establish a result regarding $\mathrm{e}_{\mathrm{HK}}(I, R) - \mathrm{e}_{\mathrm{HK}}(J, R)$ for $\mathfrak{m}$-primary ideals $I \subsetneq J$. To be specific, Theorem 3.1 shows that there exists a real number $\delta > 0$, depending only on I, such that $\mathrm{e}_{\mathrm{HK}}(I, R) - \mathrm{e}_{\mathrm{HK}}(J, R)$ is either 0 or $\geqslant \delta$, for all J containing I. This result will be used to prove Theorem 4.1.

In Section 4, using Theorem 3.1, we prove that, under mild conditions (e.g., when R is excellent), $\mathrm{r}(R) > 0$ if and only if R is F-rational; see Theorem 4.1. This justifies our choice of the term 'F-rational signature' for $\mathrm{r}(R)$.

In Section 5, we establish results on how the F-rational signature behaves under deformation, local flat extension and localization; see Theorem 5.1, Theorem 5.6, Theorem 5.7 and Theorem 5.9.

In Section 6, we define *phantom F-rational signature*, denoted $\mathrm{r}'(R)$, using modules of finite length and of finite phantom projective dimension. Making use of a result in [HoY], we show that (under mild conditions, e.g., if R is excellent) R is F-rational if and only if $\mathrm{r}'(R) > 0$; see Theorem 6.3.

In the final section, we calculate the F-rational signature of some specific rings of interest.

1. Background and preliminary results

Let R be a Noetherian ring of prime characteristic p. The Frobenius homomorphism $F : R \to R$ is defined by $r \mapsto r^p$ for $r \in R$. For any $e \in \mathbb{N}$, we have the iterated Frobenius homomorphism $F^e : R \to R$ by $r \mapsto r^q$ for $r \in R$, where $q = p^e$. From now on, q will be used to denote the value p^e for various $e \in \mathbb{N}$ in the context. Similarly, we use $Q = p^E, q_0 = p^{e_0}, q' = p^{e'}, q'' = p^{e''}$, etc. to denote varying powers of p.

Let M be an R-module. For any $e \geqslant 0$, we define a left R-module structure on the set M by $r \cdot m = r^{p^e} m$ for any $r \in R$ and $m \in M$. We keep the original right R-module structure on M. We denote the resulting R-R-bimodule by ${}^{e}M$: we have $r \cdot m = m \cdot r^{p^e}$, which is equal to $r^q m$ in the original M.

If R is reduced, then ${}^{e}R$, as a left R-module, is isomorphic to $R^{1/q}$. (One way to interpret $R^{1/q}$ is as follows: Let $P_1, \ldots, P_n$ be the minimal primes of R, so that R is, up to natural isomorphism, a subring of $\prod_{i=1}^n R/P_i \subseteq \prod_{i=1}^n \kappa(P_i) \subseteq \prod_{i=1}^n \kappa(P_i)^{1/q}$, in which $\kappa(P_i)$ is the fraction field of R/P_i. Then $R^{1/q} = \{a \in \prod_{i=1}^n \kappa(P_i)^{1/q} \mid a^q \in R\}$.)

We use $\lambda^l(-), \lambda^r(-)$ to denote the left and right lengths of a bimodule. It is easy to see that $\lambda^l({}^{e}M) = q^{\alpha(R)}\lambda^r({}^{e}M) = q^{\alpha(R)}\lambda(M)$ for any finite length R-module M, where $\alpha(R) = \log_p[k : k^p]$.

We recall that R is F-finite if ${}^1R$ is a finitely generated left R-module. If this is the case, it is easy to see that ${}^eM$ is a finitely generated left R-module for every $e \in \mathbb{N}$ and for every finitely generated R-module M.

For an ideal I of R, we denote by $I^{[q]}$ the ideal generated by $\{r^q \mid r \in I\}$. Then $R/I \otimes_R {}^eM \cong {}^e(M/I^{[q]}M) \cong {}^eM \otimes_R R/I^{[q]}$ for every R-module M and every $e \in \mathbb{N}$. (Note that if M is an R-S-bimodule and N is an S-T-bimodule then $M \otimes_S N$ is defined and is naturally an R-T-bimodule.)

A very important concept in studying rings of characteristic p is tight closure. Tight closure was first studied and developed by Hochster and Huneke in the 1980s.

Definition 1.1 ([HH1])**.** Let R be a Noetherian ring of characteristic p and $N \subseteq M$ be R-modules. The tight closure of N in M, denoted by N^*_M, is defined as follows: An element $x \in M$ is said to be in N^*_M if there exists an element $c \in R^\circ$ such that $x \otimes c \in N^{[q]}_M \subseteq M \otimes_R {}^eR$ for all $e \gg 0$, where R° is the complement of the union of all minimal primes of the ring R and $N^{[q]}_M$ denotes the right R-submodule of $M \otimes_R {}^eR$ generated by $\{x \otimes 1 \in M \otimes_R {}^eR \mid x \in N\}$. The element $x \otimes 1 \in M \otimes_R {}^eR$ is denoted by $x^{p^e}_M = x^q_M$ and $M \otimes_R {}^eR$ is denoted by $F^e(M)$.

With the above notation, the Frobenius closure of N in M, denoted N^F_M, is defined as follows: $x \in N^F_M$ if and only if $x^q_M \in N^{[q]}_M$ for some $q = p^e$ (thus for all $q = p^e \gg 1$).

Given R-modules $N \subseteq M$, N^*_M/N, as an R-submodule of M/N, is exactly $0^*_{M/N}$; we say that N is tightly closed (in M) if $N^*_M = N$. It is clear that $N^F_M \subseteq N^*_M$.

Definition 1.2 ([FW], [HH2, 4.1], [Hu, 3.11])**.** A Noetherian ring R of characteristic p is said to be F-rational if every parameter ideal of R (i.e., every (proper) ideal whose height is equal to its minimal number of generators) is tightly closed. (The ideal 0 is a parameter ideal; thus, an F-rational ring is reduced.)

Remark 1.3. Further assume that $(R, \mathfrak{m}, k)$ is local, of dimension d. Then R is F-rational if and only if all ideals generated by a system of parameters (s.o.p.) are tightly closed. The 'only if' direction is clear since every ideal generated by a s.o.p. is a parameter ideal. Conversely, for every parameter ideal $I = (x_1, \dots, x_i)$ with height i, there exist $x_{i+1}, \dots, x_d$ such that $x_1, \dots, x_i, x_{i+1}, \dots, x_d$ form a s.o.p., hence $I^* \subseteq \bigcap_{n\in\mathbb{N}}(x_1, \dots, x_i, x^n_{i+1}, \dots, x^n_d)^* = \bigcap_{n\in\mathbb{N}}(x_1, \dots, x_i, x^n_{i+1}, \dots, x^n_d) = (x_1, \dots, x_i) = I$.

Theorem 1.4 ([HH2, Theorem 4.2])**.** *For a Noetherian ring of characteristic p the following hold:*

(1) *An F-rational ring is normal.*
(2) *An F-rational ring that is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay.*
(3) *A local ring $(R, \mathfrak{m})$ that is a homomorphic image of a Cohen-Macaulay ring is F-rational if and only if it is equidimensional and the ideal generated by one system of parameters is tightly closed.*
(4) *Let R be an F-rational ring that is a homomorphic image of a Cohen-Macaulay ring. Then every localization of R is F-rational.*
(5) *If $(R, \mathfrak{m})$ is a homomorphic image of a Cohen-Macaulay ring and $x \in \mathfrak{m}$ is a non-zero-divisor such that R/xR is F-rational, then R is F-rational.*

The notion of test element is very important in studying tight closure.

Definition 1.5 ([HH1], [HH2]). Let R be a Noetherian ring of characteristic p, $c \in R^\circ$ and $e_0 \geqslant 0$. We say that c is a p^{e_0}-weak test element if, for every finitely generated R-module M and all $x \in 0^*_M$, we have $0 = x \otimes c \in M \otimes_R {}^eR$ for all $e \geqslant e_0$. Also, we say that $c \in R^\circ$ is a parameter p^{e_0}-weak test element if $cx^q \in I^{[q]}$ for every ideal I generated by a system of parameters, all $x \in I^*$, and all $q = p^e$ with $e \geqslant e_0$. A p^{e_0}-weak test element c is said to be locally (completely) stable if c is a p^{e_0}-weak test element for (the completion of) every local ring of R. A (parameter, locally stable, completely stable) weak test element means a (parameter, locally stable, completely stable) p^e-weak test element for some $e \geqslant 0$. A (parameter, locally stable, completely stable) test element means a p^0-weak (parameter, locally stable, completely stable) test element.

Clearly, every (weak) test element is a parameter (weak) test element. The existence of (weak) test elements is established in the following

Theorem 1.6 (Existence of test elements, [HH2]). *Let R be a reduced algebra of finite type over an excellent local ring $(B, \mathfrak{m})$ of characteristic p. Let $c \in R^\circ$ be such that $R[c^{-1}]$ is F-regular and Gorenstein (e.g., $R[c^{-1}]$ is regular, so such c exists). Then c has a power that is a completely stable test element for R.*

If we remove the assumption that R is reduced, then a completely stable weak test element exists.

In particular, if $(R, \mathfrak{m})$ is a reduced excellent local ring of characteristic p, then every element in the defining ideal of the singular locus of R has a power which is a completely stable test element for R. By [Ku], an F-finite ring is excellent. Therefore completely stable test elements always exist for reduced F-finite local rings of characteristic p.

One of the major questions in tight closure theory *was* whether tight closure commutes with localization, i.e., whether $(N^*_M)_P = (N_P)^*_{M_P}$ for every $P \in \operatorname{Spec}(R)$ and every finitely generated R-module $N \subseteq M$. The question has been answered by Brenner and Monsky in [BM], which shows that tight closure does not commute with localization in general. One reason why the question was hard is that the definition of tight closure involves infinitely many equations (i.e., we need $0 = x \otimes c \in M \otimes_R {}^eR$ for all $q \gg 0$ to make sure that an element $x \in M$ is in 0^*_M). There is a notion of *test exponent*, which has been introduced in [HH4] to avoid the infinitely many equations in the definition of tight closure.

Definition 1.7 ([HH4]). Let R be a (reduced) Noetherian ring of prime characteristic p, $c \in R^\circ$ (a test element), and $N \subseteq M$ R-modules. We say that $q_0 = p^{e_0}$ is a test exponent for c and $N \subseteq M$ if, for any $x \in M$, we have $x \in N^*_M$ whenever $x \otimes c \in N^{[q]}_M \subseteq F^e(M)$ for one single $q \geqslant q_0$.

If there exists a test exponent for a locally stable test element $c \in R^\circ$ and finitely generated R-modules $N \subseteq M$, then the tight closure of N in M commutes with localization. This result is implicit in [McD] and is explicitly stated in [HH4, Proposition 2.3]. Moreover, Hochster and Huneke showed in [HH4] that the converse is true.

Theorem 1.8 ([HH4])**.** *Let R be a (reduced) Noetherian ring of prime characteristic p with a given locally stable test element c, and $N \subseteq M$ finitely generated R-modules. Assume that the tight closure of N in M commutes with localization. Then there exists a test exponent for c and $N \subseteq M$.*

In particular, if $\lambda(M/N) < \infty$, then there exists a test exponent for c and $N \subseteq M$, since tight closure commutes with localization in this case (cf. [HH1, Proposition 8.9]*).*

Next we review the Hilbert-Kunz multiplicity.

Theorem 1.9. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic p with* $\dim(R) = d$ *and M a finitely generated R module. Then (with $q = p^e$)*

(1) For any R-module L with $\lambda_R(L) < \infty$, the limit

$$\lim_{e \to \infty} \frac{\lambda^r(L \otimes_R {}^eM)}{q^d}$$

exists by [Se]*. (The statement in* [Se, Page 278, Theorem] *is stronger, but its proof requires F-finiteness. However, the result quoted here does not need F-finiteness, as it always reduces to the F-finite case.)*

(2) In particular, if $L = R/I$, with I an $\mathfrak{m}$-primary ideal, the limit

$$\lim_{e \to \infty} \frac{\lambda_R^r(R/I \otimes_R {}^eM)}{q^d} = \lim_{e \to \infty} \frac{\lambda_R(M/I^{[q]}M)}{q^d}$$

exists. This particular case was first proved in [Mo]*.*

Notation 1.10. Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p with $\dim(R) = d$, L and M finitely generated R-modules with $\lambda_R(L) < \infty$.

(1) We denote $\mathrm{e_{HK}}(L, M) := \lim_{e\to\infty} \frac{\lambda_R^r(L \otimes_R {}^eM)}{q^d}$, which is positive if and only if $\dim(M) = d$.
(2) In the case where $L = R/I$ with I an $\mathfrak{m}$-primary ideal, we usually write $\mathrm{e_{HK}}(L, M)$ as $\mathrm{e_{HK}}(I, M)$, which is called the Hilbert-Kunz multiplicity of M with respect to I. In particular, $\mathrm{e_{HK}}(I, M) > 0$ if and only if $\dim(M) = d$.

The following result is referred to as 'the length criterion for tight closure' in [HH1]. In [HH1, Theorem 8.17], more general results are proved.

Theorem 1.11 ([HH1, Theorem 8.17])**.** *Let $(R, \mathfrak{m})$ be a local Noetherian ring, let M and $K \subseteq L$ be R-modules such that* $\dim(M) = \dim(R)$ *and $\lambda(L) < \infty$, and let $I \subseteq J$ be $\mathfrak{m}$-primary ideals of R.*

(1) If $K \subseteq 0^*_L$, then $\mathrm{e_{HK}}(L, M) = \mathrm{e_{HK}}(L/K, M)$*. In particular, if $J \subseteq I^*$, then* $\mathrm{e_{HK}}(I, M) = \mathrm{e_{HK}}(J, M)$*.*

(2) Conversely, assume that R is an analytically unramified, quasi-unmixed ring with a completely stable test element (e.g., $(R, \mathfrak{m})$ is a complete domain). If $\mathrm{e_{HK}}(L, R) = \mathrm{e_{HK}}(L/K, R)$*, then $K \subseteq 0^*_L$. In particular,* $\mathrm{e_{HK}}(I, R) = \mathrm{e_{HK}}(J, R)$ *implies $J \subseteq I^*$.*

Next, we review F-signature, which was first introduced and studied in [HL] by C. Huneke and G. Leuschke for F-finite rings.

Definition 1.12. Let $(R, \mathfrak{m}, k)$ be an F-finite local ring with $d = \dim R$ and M a finitely generated R-module. For each $e \in \mathbb{N}$, write ${}^e M \cong R^{a_e} \oplus M_e$ as left R-modules, where M_e has no non-zero free direct summand.

(1) Denote a_e by $\#({}^e M, R)$ and $\alpha(R) = \log_p [k : k^p] < \infty$.
(2) We define $\mathrm{s}(M) = \lim_{e\to\infty} \frac{\#({}^e M, R)}{q^{\alpha(R)+d}}$, whose existence is due to [Tu].

In particular, we call $\mathrm{s}(R)$ the F-signature of R (see [HL]).

In [Yao], a definition of F-signature is given for all Noetherian local rings of prime characteristic p and it is equivalent to Definition 1.12 when R is F-finite.

Definition 1.13. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p and M a finitely generated R-module. Let $E = E_R(k)$ be the injective hull of k and let $\psi : k \to E$ and $\phi : E \to E/k$ be such that $0 \to k \xrightarrow{\psi} E \xrightarrow{\phi} E/k \to 0$ is exact.

(1) Denote $\#({}^e M) = \lambda^r(\ker(\phi \otimes_R 1_{{}^e M})) = \lambda^r(\mathrm{Image}(\psi \otimes_R 1_{{}^e M}))$ for all $e \in \mathbb{N}$.
(2) We define $\mathrm{s}(M) := \lim_{e\to\infty} \frac{\#({}^e M)}{q^{\dim(R)}}$, whose existence is proved in [Tu].

In particular, we call $\mathrm{s}(R)$ the F-signature of R.

Theorem 1.14 ([HL], [AL])**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic p. Then the following are true:*

(1) If $\mathrm{s}(R) > 0$, then R is an F-regular and Cohen-Macaulay domain [HL].
(2) If R is excellent (e.g., F-finite), then $\mathrm{s}(R) > 0 \iff R$ is strongly F-regular; see [AL]. *Although the argument in* [AL] *addresses the F-finite case only, it can be generalized to the case where $(R, \mathfrak{m})$ is excellent. We sketch an alternate proof of this in Remark 3.6 assuming only that $(R, \mathfrak{m})$ is excellent.*
(3) We have $\mathrm{e_{HK}}(I, R) - \mathrm{e_{HK}}(J, R) \geqslant \lambda_R(J/I)\,\mathrm{s}(R)$ for any two $\mathfrak{m}$-primary ideals $I \subseteq J$ of R (see [HL]*). Therefore*

$$\mathrm{s}(R) \leqslant \inf\{\mathrm{e_{HK}}(I_1) - \mathrm{e_{HK}}(I_2) \mid I_1 \subset I_2, \sqrt{I_1} = \mathfrak{m}, I_2/I_1 \cong k\}.$$

(4) The inequality $(\mathrm{e}(R) - 1)(1 - \mathrm{s}(R)) \geqslant \mathrm{e_{HK}}(R) - 1$ is proved in [HL]*. Hence, $\mathrm{s}(R) \geqslant 1 \implies R$ is regular $\implies \mathrm{s}(R) = 1$.*
(5) Moreover, in [HL] *it is shown that if R is Gorenstein then $\mathrm{s}(R) = \mathrm{e_{HK}}((\underline{x}), R) - \mathrm{e_{HK}}((\underline{x}, u), R)$ for any system of parameters $\underline{x}$ and $u \in ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})$.*

Remark 1.15. In [WY], K.-i. Watanabe and K. Yoshida asked whether $\inf\{\mathrm{e_{HK}}(I_1) - \mathrm{e_{HK}}(I_2) \mid I_1 \subset I_2, \sqrt{I_1} = \mathfrak{m}, I_2/I_1 \cong k\}$ is equal to $\mathrm{s}(R)$. This is proved affirmatively for approximately Gorenstein rings by Polstra and Tucker [PT].

The next result is used in Section 5. The exact statement of the following theorem can be found in [HH2, Theorem 7.10], which refers readers to a more general result in [Mat1, 20.F].

Theorem 1.16. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local flat ring homomorphism. If $x_1, x_2, \ldots, x_t$ form a regular sequence on $S/\mathfrak{m}S$, then they form a regular sequence on S and $R \to S/(x_1, x_2, \ldots, x_t)S$ is again a (faithfully) flat local homomorphism.*

2. Defining F-rational signature

We have defined the F-rational signature $\mathrm{r}(R)$ in Definition 0.1. Now we define the more general notion $\mathrm{r}(M)$ (and give some equivalent ways of defining it), as follows.

Definition 2.1. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p and M a finitely generated R-module. Let s.o.p. stand for system of parameters, and make the convention that $\mathrm{e_{HK}}(J, M) = 0$ when $J = R$. Define

$$\mathrm{r}_R(M) = \inf\{\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}(J, M) \mid \underline{x} \text{ is a s.o.p. and } (\underline{x}) \subsetneq J\}.$$

If no confusion arises, we may simply write $\mathrm{r}_R(M)$ as $\mathrm{r}(M)$. In particular, $\mathrm{r}(R)$ is called the F-rational signature of R.

In the definition above, $\underline{x}$, a priori, runs over all (full) systems of parameters of R. It is also straightforward to see that

$$\begin{aligned}\mathrm{r}_R(M) &= \inf\{\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}((\underline{x}, u), M) \mid \underline{x} \text{ is a s.o.p. and } ((\underline{x}) :_R u) = \mathfrak{m}\} \\ &= \inf\{\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}((\underline{x}, v), M) \mid \underline{x} \text{ is a s.o.p. and } v \notin (\underline{x})\}.\end{aligned}$$

Observation 2.2. Let $(R, \mathfrak{m}, k)$ and M be as in Definition 2.1. Denote the $\mathfrak{m}$-adic completions of R and M by $\widehat{R}$ and $\widehat{M}$ respectively. We observe:

(1) The $\mathfrak{m}$-primary ideals of R naturally correspond to the $\mathfrak{m}\widehat{R}$-primary ideals of $\widehat{R}$, with the corresponding ideals sharing the same Hilbert-Kunz multiplicities and the same number of generaters over R and $\widehat{R}$. Hence the ideals of R generated by systems of parameters of R naturally correspond to the ideals of $\widehat{R}$ generated by systems of parameters of $\widehat{R}$. Consequently, $\mathrm{r}_R(M) = \mathrm{r}_{\widehat{R}}(\widehat{M})$. (See Theorem 5.7(3) for a more general statement.)
(2) If $\dim(M) < \dim(R)$, then $\mathrm{r}(M) = 0$ because $\mathrm{e_{HK}}(I, M) = 0$ for any $\mathfrak{m}$-primary ideal I.
(3) Suppose that R is a domain (e.g., R is F-rational) and say that the torsion-free rank of M is n (i.e., $M_P \cong R_P^n$ when $P = 0 \in \operatorname{Spec}(R)$). Then, by the additivity of Hilbert-Kunz multiplicity with respect to short exact sequences [Se], $\mathrm{e_{HK}}(J, M) = n\,\mathrm{e_{HK}}(J, R)$ for any $\mathfrak{m}$-primary ideal J of R. This implies that $\mathrm{r}(M) = n\,\mathrm{r}(R)$ from the definitions of $\mathrm{r}(M)$ and $\mathrm{r}(R)$. Obviously, $n > 0$ if and only if $\dim(M) = \dim(R)$.
(4) If $\widehat{R}$ is not F-rational (in particular, if R is not F-rational), then there exists a system of parameters $\underline{x}$ of R (hence, of $\widehat{R}$) and $u \in (((\underline{x})\widehat{R})^* \cap R) \setminus (\underline{x})R$, which gives $\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}((\underline{x}, u), M) = 0$. This implies that $\mathrm{r}(M) = 0$ for any finitely generated R-module M.
(5) If R is not a normal Cohen-Macaulay domain, then $\mathrm{r}(M) = 0$ for any M. Indeed, if R is not a normal domain, then R can not be F-rational. If R is not Cohen-Macaulay, then $\widehat{R}$ is not Cohen-Macaulay, so can not be F-rational. Hence, $\mathrm{r}(M) = 0$ for any finitely generated R-module M, by part (4) above.

The following result relates $\mathrm{e_{HK}}(I, M) - \mathrm{e_{HK}}(J, M)$, in which $I \subseteq J$ and I is generated by a system of parameters of a Cohen-Macaulay local ring R and M is a maximal

Cohen-Macaulay R-module, to the limit of a sequence determined intrinsically by the image of the natural and injective map $J/I \to \varinjlim_{\underline{x} \text{ s.o.p.}} R/(\underline{x}) \cong \mathrm{H}_{\mathfrak{m}}^{\dim(R)}(R)$.

Proposition 2.3. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local Cohen-Macaulay ring of prime characteristic p with $\dim R = d$ and let M be a finitely generated maximal Cohen-Macaulay module. Denote $H = \mathrm{H}_{\mathfrak{m}}^d(R)$.*

Let L be an R-module and $\psi : L \to H$ an R-linear map such that $\lambda_R(\psi(L)) < \infty$. There is an induced bimodule homomorphism $\psi \otimes_R 1_{{}^eM} : L \otimes_R {}^eM \to H \otimes_R {}^eM$ for every $e \in \mathbb{N}$. Then

$$\lim_{e\to\infty} \frac{\lambda^r(\mathrm{Image}(\psi \otimes_R 1_{{}^eM}))}{q^d} = \mathrm{e_{HK}}(I, M) - \mathrm{e_{HK}}(J, M)$$

for any ideals $I \subseteq J$ such that $I = (\underline{x})$ is generated by a system of parameters and $J/I \cong \psi(L)$. Moreover, such ideals $I \subseteq J$ always exist.

Conversely, for any $\mathfrak{m}$-primary ideals $I \subseteq J$ such that I is generated by a system of parameters, there exist an R-module L and an R-linear (inclusion) map $\psi : L \to H$ such that $\psi(L) \cong J/I$ and $\mathrm{e_{HK}}(I, M) - \mathrm{e_{HK}}(J, M) = \lim_{e\to\infty} \frac{\lambda^r(\mathrm{Image}(\psi \otimes_R 1_{{}^eM}))}{q^d}$.

Proof. To prove the claim in the first paragraph, we may simply replace L by $\psi(L)$ and, hence, assume $L \subseteq H$, $\lambda_R(L) < \infty$, where ψ is the inclusion map. There exists a system of parameters $\underline{x} = x_1, x_2, \ldots, x_d$ of R such that $R/(\underline{x}) \cong K \subseteq H$ and $L \subseteq K$. Say L corresponds to $(\underline{x}, \underline{u})/(\underline{x}) \subseteq R/(\underline{x})$ under the isomorphism $R/(\underline{x}) \cong K$, where $\underline{u} = u_1, u_2, \ldots, u_r$.

Let $\phi : K \to H$ be the inclusion map. Then there is an induced bimodule homomorphism $\phi \otimes_R 1_{{}^eM} : K \otimes_R {}^eM \to H \otimes_R {}^eM$ for every $e \in \mathbb{N}$. As M is Cohen-Macaulay, we see that, for every $e \in \mathbb{N}$, ${}^eM$ is a big (i.e., not necessarily finitely generated) Cohen-Macaulay left R-module. From this it is straightforward to see that the maps $\phi \otimes_R 1_{{}^eM} : K \otimes_R {}^eM \to H \otimes_R {}^eM$ are injective for all $e \in \mathbb{N}$.

Hence, by our set-up, for all $e \in \mathbb{N}$ we have

$$\begin{aligned}
\lambda^r(\mathrm{Image}(\psi \otimes_R 1_{{}^eM})) &= \lambda^r(\mathrm{Image}(L \otimes_R 1_{{}^eM} \to K \otimes_R 1_{{}^eM})) \\
&= \lambda^r\left(\mathrm{Image}\left(\frac{(\underline{x}, \underline{u})}{(\underline{x})} \otimes 1_{{}^eM} \to \frac{R}{(\underline{x})} \otimes 1_{{}^eM}\right)\right) \\
&= \lambda^r\left(\frac{R}{(\underline{x})} \otimes_R {}^eM\right) - \lambda^r\left(\frac{R}{(\underline{x}, \underline{u})} \otimes_R {}^eM\right) \\
&= \lambda\left(\frac{M}{(\underline{x})^{[q]}M}\right) - \lambda\left(\frac{M}{(\underline{x}, \underline{u})^{[q]}M}\right),
\end{aligned}$$

which gives the existence of

$$\lim_{e\to\infty} \frac{\lambda^r(\mathrm{Image}(\psi \otimes_R 1_{{}^eM}))}{q^d} = \lim_{e\to\infty} \frac{\lambda\left(\frac{M}{(\underline{x})^{[q]}M}\right) - \lambda\left(\frac{M}{(\underline{x}, \underline{u})^{[q]}M}\right)}{q^d},$$

and this is $\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}((\underline{x}, \underline{u}), M)$.

The other direction follows immediately by embedding $J/I \subseteq R/I$ to $\mathrm{H}_{\mathfrak{m}}^d(R)$ just as in the above proof. $\square$

Concerning F-rational signature, Proposition 2.3 shows the following:

Theorem 2.4. *Let $(R, \mathfrak{m}, k)$ be a Noetherian Cohen-Macaulay local ring of characteristic p with $\dim(R) = d$ and M a finitely generated maximal Cohen-Macaulay R-module. Then*

(1) $\mathrm{r}_R(M) = \inf\left\{ \lim_{e\to\infty} \frac{\lambda^r(\mathrm{Image}(\psi \otimes_R 1_{^eM}))}{q^d} \,\middle|\, 0 \to k \xrightarrow{\psi} \mathrm{H}^d_{\mathfrak{m}}(R) \text{ is exact}\right\}$;

(2) For any fixed system of parameters $\underline{x}$, we always have

$$\mathrm{r}_R(M) = \inf\{\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}((\underline{x}, u), M) \,|\, ((\underline{x}) :_R u) = \mathfrak{m}\}.$$

Proof. (1). Apply Proposition 2.3 to all injective maps $\psi : k \to \mathrm{H}^d_{\mathfrak{m}}(R)$.

(2). For every exact sequence $0 \to k \xrightarrow{\psi} \mathrm{H}^d_{\mathfrak{m}}(R)$, $L = \mathrm{Image}(\psi)$ is contained in $(0 :_{\mathrm{H}^d_{\mathfrak{m}}(R)} \mathfrak{m})$, the socle of $\mathrm{H}^d_{\mathfrak{m}}(R)$. Say $R/(\underline{x}) \cong K \subseteq \mathrm{H}^d_{\mathfrak{m}}(R)$. Then, since R is Cohen-Macaulay, $L \subseteq (0 :_{\mathrm{H}^d_{\mathfrak{m}}(R)} \mathfrak{m}) = (0 :_K \mathfrak{m})$, i.e., ψ factors through K. Hence, $\lim_{e\to\infty} \frac{\lambda^r(\mathrm{Image}(\psi \otimes_R 1_{^eM}))}{q^d} = \mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}((\underline{x}, u), M)$ for some $u \in ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})$ by Proposition 2.3. Now (2) follows from (1). □

It turns out that Theorem 2.4(2) remains true without the Cohen-Macaulay assumption on R or M.

Theorem 2.5. *Suppose that $(R, \mathfrak{m}, k)$ is a Noetherian local ring of characteristic p and M is a finitely generated R-module. Say $\dim(R) = d$. Assume R is an equidimensional homomorphic image of a Cohen-Macaulay ring or $\widehat{R}$ is equidimensional (e.g., R is excellent and equidimensional). Then for any fixed system of parameters $\underline{x} = x_1, x_2, \ldots, x_d$, we always have*

$$\begin{aligned}\mathrm{r}_R(M) &= \inf\{\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}((\underline{x}, u), M) \,|\, ((\underline{x}) :_R u) = \mathfrak{m}\} \\ &= \inf\{\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}((\underline{x}, v), M) \,|\, v \notin (\underline{x})\} \\ &= \inf\{\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}(J, M) \,|\, (\underline{x}) \subsetneq J\},\end{aligned}$$

Proof. As the claim is unaffected if we pass to $\widehat{R}$ and $\widehat{M}$, the $\mathfrak{m}$-adic completions of R and M, we may assume that R is an equidimensional homomorphic image of a Cohen-Macaulay ring without loss of generality.

If $(\underline{x}) \subsetneq (\underline{x})^*$, then there exists $u \in (\underline{x})^* \cap ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})$ giving $\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}((\underline{x}, u), M) = 0$. Thus, $\mathrm{r}(M) = 0$ and the claim holds.

Assume $(\underline{x}) = (\underline{x})^*$, which implies that R is F-rational and, hence, a Cohen-Macaulay normal domain; cf. Theorem 1.4. Say that M has torsion-free rank n over R. Then $\mathrm{r}(M) = n\,\mathrm{r}(R)$ (cf. Observation 2.2(3)). Now apply Theorem 2.4(2) and we get $\mathrm{r}(R) = \inf\{\mathrm{e_{HK}}((\underline{x}), R) - \mathrm{e_{HK}}((\underline{x}, u), R) \,|\, u \in ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})\}$. Consequently,

$$\begin{aligned}\mathrm{r}(M) = n\,\mathrm{r}(R) &= \inf\{n\,\mathrm{e_{HK}}((\underline{x}), R) - n\,\mathrm{e_{HK}}((\underline{x}, u), R) \,|\, u \in ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})\} \\ &= \inf\{\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}((\underline{x}, u), M) \,|\, ((\underline{x}) :_R u) = \mathfrak{m}\}\end{aligned}$$

and the proof is complete. □

3. Drops in the Hilbert-Kunz multiplicity

By the definition of the Hilbert-Kunz multiplicity, $\mathrm{e_{HK}}(I, M) \geqslant \mathrm{e_{HK}}(J, M)$ for any $\mathfrak{m}$-primary ideals $I \subseteq J$. The following result shows how the Hilbert-Kunz multiplicity decreases when the ideal increases from I to J.

Theorem 3.1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p and let $N \subseteq L$ be R-modules such that $\lambda(L/N) < \infty$. Then*

1. *There exists a positive real number δ such that, for any R-submodule K with $N \subseteq K \subseteq L$ and for any finitely generated R-module M, $\mathrm{e_{HK}}(L/N, M) - \mathrm{e_{HK}}(L/K, M)$ is either 0 or $\geqslant \delta$.*
2. *In case $L = R$ and $N = I$ is an $\mathfrak{m}$-primary ideal of R, there exists a positive real number δ such that, for any ideal $J \supseteq I$ and for any finitely generated R-module M, $\mathrm{e_{HK}}(I, M) - \mathrm{e_{HK}}(J, M)$ is either 0 or $\geqslant \delta$.*

Proof. It is enough to prove part (1), and we may assume that R is complete without loss of generality. Say $\min(R) = \{P_1, P_2, \ldots, P_n\}$ and say $\dim(R/P_i) = \dim(R)$ exactly when $1 \leqslant i \leqslant s$, for some $s \leqslant n$. Since Hilbert-Kunz multiplicity is additive with respect to short exact sequences (see [Se]), the associativity formula holds. Thus, for any R-submodule $K \subseteq L$ and for any finitely generated R-module M,

$$\begin{aligned}
&\mathrm{e_{HK}}(L/N, M) - \mathrm{e_{HK}}(L/K, M) \\
&\quad = \sum_{i=1}^{s} \lambda_{R_{P_i}}(M_{P_i})\big(\mathrm{e_{HK}}(L/N, R/P_i) - \mathrm{e_{HK}}(L/K, R/P_i)\big) \\
&\quad = \sum_{i=1}^{s} \lambda_{R_{P_i}}(M_{P_i})\big(\mathrm{e_{HK}}(L/(N + P_iL), R/P_i) - \mathrm{e_{HK}}(L/(K + P_iL), R/P_i)\big).
\end{aligned}$$

Therefore it suffices to prove the desired result under the assumption that R is a complete local domain and $M = R$. Now the proof follows immediately from the following Theorem 3.2. □

Theorem 3.2. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p such that its $\mathfrak{m}$-adic completion $\widehat{R}$ is a domain. Assume that there is a common (weak) test element for R and $\widehat{R}$ (e.g., R is excellent). Let $N \subseteq L$ be R-modules such that $\lambda(L/N) < \infty$. Then there exists a positive real number δ such that, for any R-submodule K with $N \subseteq K \subseteq L$, exactly one of the following happens:*

1. *If $K \subseteq N^*_L$, then $\mathrm{e_{HK}}(L/N, R) - \mathrm{e_{HK}}(L/K, R) = 0$.*
2. *If $K \not\subseteq N^*_L$, then $\mathrm{e_{HK}}(L/N, R) - \mathrm{e_{HK}}(L/K, R) \geqslant \delta$.*

Proof. (1). This was proved in [HH1, Theorem 8.17] (cf. Theorem 1.11) without any assumption on R at all.

(2) This part is mostly implicit in the proof of [HH1, Theorem 8.17] in light of the existence of a test exponent by [HH4]. Nevertheless, we present a proof here for completeness. As R and $\widehat{R}$ share a common (weak) test element, we see that $K \not\subseteq N^*_L$ over R if and only if $K \otimes \widehat{R} \not\subseteq (N \otimes \widehat{R})^*_{L \otimes \widehat{R}}$ over $\widehat{R}$. Therefore, we may assume that

$R = \widehat{R}$ is complete (hence a complete domain) without loss of generality. We may further assume $N = 0$.

Our complete local domain $(R, \mathfrak{m}, k)$ is a module-finite and torsion-free extension of a complete regular local domain A with the same coefficient field. For every q, we can form $R[A^{1/q}] \subseteq R^{1/q}$. Moreover, there exists q' such that $S = R[A^{1/q'}]$ is generically smooth over $A^{1/q'}$ (meaning the fraction field of S is separable over the fraction field of $A^{1/q'}$, given that S is module-finite over $A^{1/q'}$)[1]. By [HH1, Section 6], $S[A^{1/qq'}] \cong S \otimes_{A^{1/q'}} A^{1/qq'}$ is flat over $S = R[A^{1/q'}]$ and there exists $c \in A^\circ$ such that $cS^{1/q} \subseteq S[A^{1/qq'}]$ for all q. The fact that $R \subseteq S \subseteq R^{1/q'}$ shows that S is integral over R and hence integral over A, which implies $(IS) \cap A \subseteq I_A^* = I$ for every ideal I of A.[2]

For easy identification, we denote the maximal ideals of the complete local domains A and S by $\mathfrak{m}_A$ and $\mathfrak{m}_S$ respectively. Since S is integral over A (and S is local), there exists q'' such that $\mathfrak{m}_S^{[q'']} \subseteq \mathfrak{m}_A S$.

Pick $c' \in A^\circ$ to be a locally stable test element for R (which exists by Theorem 1.6). Then $cc' \in A^\circ$ is also a locally stable test element for R. By [HH4] (see Theorem 1.8 and observe the fact that $\lambda(L) < \infty$), there exists a test exponent, say $q''' = p^{e'''}$, for cc' and $N = 0 \subseteq L$. (More generally, by a result in [HoY], there exists a test exponent for cc' and $N \subseteq L$ as long as L/N is Artinian over R.)

Let $\delta = (1/q''q''')^d > 0$, in which $d = \dim(R) = \dim(A)$, and let K be an arbitrary R-submodule of L such that $K \nsubseteq 0_L^*$. We need to show $\mathrm{e_{HK}}(L, R) - \mathrm{e_{HK}}(L/K, R) \geqslant \delta$.

Choose $x \in K \backslash 0_L^*$. By our choice of test exponent q''', we have $0 \neq x \otimes c \in L \otimes_R {}^{e'''}S$. Indeed, on the contrary, suppose that $0 = x \otimes c \in L \otimes_R {}^{e'''}S$. Then $0 = x \otimes c \in L \otimes_R {}^{e'''}(R^{1/q'}) \cong L \otimes_R {}^{e'''}R \otimes_R R^{1/q'}$, which would imply $x \otimes c \in 0^F_{F_R^{e'''}(L)} \subseteq 0^*_{F_R^{e'''}(L)}$. Consequently, $0 = x \otimes cc' \in F_R^{e'''}(L)$, and, therefore, $x \in 0_L^*$, a contradiction.

At this point, the desired result that $\mathrm{e_{HK}}(L, R) - \mathrm{e_{HK}}(L/K, R) \geqslant \delta$ follows immediately from Theorem 3.3 below with $M = R$, and the proof of Theorem 3.2 will be complete once Theorem 3.3 is proved. □

Theorem 3.3. *Let $(R, \mathfrak{m}, k)$ be a Noetherian complete local domain of prime characteristic p with $\dim(R) = d$ and let $A, q', S, c \in A^\circ \subseteq R^\circ, c' \in A^\circ \subseteq R^\circ$ and q'' be as in the proof of the above Theorem 3.2. Let M be any finitely generated R-module with torsion-free rank n. Then, for any R-module $N \subset K \subseteq L$ with $\lambda(L/N) < \infty$ such that the natural image of $(K/N) \otimes c$ in $(L/N) \otimes_R {}^{e'''}S$ is not 0 (which is the case if $q''' = p^{e'''}$ is a test exponent for cc' and $N \subseteq L$, while $K \nsubseteq N_L^*$ as seen in the proof of Theorem 3.2), we have $\mathrm{e_{HK}}(L/N, M) - \mathrm{e_{HK}}(L/K, M) \geqslant \frac{n}{(q''q''')^d}$.*

Proof. Without loss of generality, we assume that $N = 0$ and $M = R$, so that $n = 1$. The assumption on e''' and K says that there is an element $x \in K$ such that $0 \neq x \otimes c \in L \otimes_R {}^{e'''}S$. Hence $\mathrm{Ann}_S(x \otimes c \in L \otimes_R {}^{e'''}S) \subseteq \mathfrak{m}_S$.

For any $Q = p^E$ (with $E \in \mathbb{N}$), let $J_Q := \{a \in A \,|\, 0 = x \otimes a \in L \otimes_R {}^{E}R\}$, which is an ideal of A. We first note that A/J_Q embeds into $K_L^{[Q]}$ via the A-linear map

[1]By the Cohen-Gabber theorem, one can choose A such that R is generically smooth over A. Also see [KS].

[2]For every $a \in (IS) \cap A$, there exists a module-finite extension S' of A such that $S' \subseteq S$ and $a \in (IS') \cap A$. By [HH3, Corollary 5.23], $(IS') \cap A \subseteq I_A^*$.

sending the class of $b \in A$ to $x \otimes b \in L \otimes_R {}^E R$. Thus $\lambda_A(A/J_Q) \leqslant \lambda_R(K_L^{[Q]})$ for every Q. For any $Q = p^E \geqslant q''q'''$, writing $Q = qq''q'''$ with $q = p^e$, we have

$$\begin{aligned}
a \in J_Q &\implies 0 = x \otimes a \in L \otimes_R {}^E R \\
&\implies 0 = x \otimes a^{1/qq''} \in L \otimes_R {}^{e'''}\big(R^{1/qq''}\big) \\
&\implies 0 = x \otimes a^{1/qq''} \in L \otimes_R {}^{e'''}\big(S^{1/qq''}\big) \\
&\implies 0 = x \otimes a^{1/qq''} c \in L \otimes_R {}^{e'''}\big(S^{1/qq''} c\big) = L \otimes_R {}^{e'''}\big(cS^{1/qq''}\big) \\
&\implies 0 = x \otimes ca^{1/qq''} \in L \otimes_R {}^{e'''}\big(S[A^{1/qq'q''}]\big) \cong L \otimes_R {}^{e'''}S \otimes_S S[A^{1/qq'q''}] \\
&\implies a^{1/qq''} \in \operatorname{Ann}_{S[A^{1/qq'q''}]}\Big(x \otimes c \in L \otimes_R {}^{e'''}\big(S[A^{1/qq'q''}]\big)\Big) \\
&\implies a^{1/qq''} \in \operatorname{Ann}_S(x \otimes c \in L \otimes_R {}^{e'''}S)S[A^{1/qq'q''}] \qquad \text{(by flatness)} \\
&\implies a^{1/qq''} \in \mathfrak{m}_S S[A^{1/qq'q''}] \\
&\implies a \in \mathfrak{m}_S^{[qq'']} S^{qq''}[A^{1/q'}] \subseteq \mathfrak{m}_S^{[qq'']} S \subseteq \mathfrak{m}_A^{[q]} S \\
&\implies a \in (\mathfrak{m}_A^{[q]} S) \cap A \subseteq \big(\mathfrak{m}_A^{[q]}\big)^*_A = \mathfrak{m}_A^{[q]} \qquad \text{(cf. proof of Theorem 3.2)},
\end{aligned}$$

which implies $J_Q \subseteq \mathfrak{m}_A^{[q]} = \mathfrak{m}_A^{[Q/q''q''']}$. Consequently,

$$\lambda_A(A/J_Q) \geqslant \lambda_A(A/\mathfrak{m}_A^{[Q/q''q''']}) = (Q/q''q''')^d \qquad \text{for all } Q \geqslant q''q'''.$$

(The last equation holds because A is regular; see [HuY, Proposition 1.5], for example.) Putting things together, we have (with $Q = p^E$)

$$\begin{aligned}
\mathrm{e}_{\mathrm{HK}}(L/N, R) - \mathrm{e}_{\mathrm{HK}}(L/K, R) &= \lim_{Q\to\infty} \frac{\lambda^r(F^E(L))}{Q^d} - \lim_{Q\to\infty} \frac{\lambda^r(F^E(L)/K_L^{[Q]})}{Q^d} \\
&= \lim_{Q\to\infty} \frac{\lambda_R(K_L^{[Q]})}{Q^d} \geqslant \lim_{Q\to\infty} \frac{\lambda_A(A/J_Q)}{Q^d} \\
&\geqslant \lim_{Q\to\infty} \frac{\left(\frac{Q}{q''q'''}\right)^d}{Q^d} = \left(\frac{1}{q''q'''}\right)^d,
\end{aligned}$$

which finishes the proof. □

The proofs of Theorem 3.2 and of Theorem 3.3 actually produce the following result, which may be viewed as a refined version of [HH1, Theorem 8.17].

Corollary 3.4. *Let $(R, \mathfrak{m}, k)$ be a Noetherian complete local domain of prime characteristic p with $\dim(R) = d$ and let A, q', S, $c \in A^\circ \subseteq R^\circ$ and q'' be as in the proof of Theorem 3.2. Let $c' \in A^\circ \subseteq R^\circ$ be such that c' is a test element for all (not necessarily finitely generated) R-modules (cf. Remark 3.5(1) below). Then, for any R-modules $N \subset K \subseteq L$ with $\lambda(K/N) < \infty$ such that the natural image of $(K/N) \otimes c$ in $(L/N) \otimes_R {}^{e'''}S$ is not 0 (which is the case if $q''' = p^{e'''}$ is a test exponent for cc' and $N \subseteq L$ while $K \nsubseteq N_L^*$, as seen in the proof of Theorem 3.2), we have $\lambda(K_L^{[Q]}/N_L^{[Q]}) \geqslant \left(\frac{Q}{q''q'''}\right)^d$ for all $Q = p^E \geqslant q''q'''$.*

Concerning the assumptions in Corollary 3.4, see the following remark on the existence of test elements that work for all (not necessarily finitely generated) R-modules and the existence of a test exponent for any given Artinian R-module.

Remark 3.5. Let $(R, \mathfrak{m}, k)$ be an excellent local ring of prime characteristic p.

(1) If R is reduced, there exists a completely stable test element that works for all (not necessarily finitely generated) R-modules. (This was proved in [El] by H. Elitzur under the assumption that R is F-finite. The general case then follows from the F-finite case via a faithfully flat extension $R \to \widehat{R} \to \widehat{R}^{\Gamma}$, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R and $\widehat{R}^{\Gamma}$ is a suitable (F-finite) Γ-extension of $\widehat{R}$. See [HH2] for details about Γ-extensions.)

(2) For any $d \in R^{\circ}$ and any R-modules $N \subseteq L$ such that L/N is Artinian, it has been shown in [HoY] that there exists a test exponent for d and $N \subseteq L$.

Finally, we end this section by pointing out an alternate proof of Theorem 1.14(2) without the F-finite assumption.

Remark 3.6. Let $(R, \mathfrak{m}, k)$ be an excellent local ring of prime characteristic p and let $E = E_R(k)$ be the injective hull of the residue field $k = R/\mathfrak{m}$. Denote by K the socle of E (so that $E \supseteq K \cong R/\mathfrak{m}$). Recall that a local ring $(R, \mathfrak{m}, k)$ is strongly F-regular if and only if $0 = 0^*_E$, i.e., $K \nsubseteq 0^*_E$ (cf. [Sm1, 7.1.2] or [LS, Proposition 2.9]).

(1) Suppose that R is complete and strongly F-regular (hence, a domain). Let A, S, $c \in A^{\circ} \subseteq R^{\circ}$, and q'' be as in the proof of Theorem 3.3. As R is strongly F-regular, there exists $q''' = p^{e'''}$ such that the natural image of $K \otimes c$ in $E \otimes_R {}^{e'''}S$ is not 0. (See the proof of Theorem 3.2 for the existence of such q'''.) Then by Theorem 3.3, we get $\lambda(K_E^{[q]}) \geqslant \left(\frac{q}{q''q'''}\right)^d$ for all $q = p^e$. Also observe that $\lambda(K_E^{[q]}) = \#({}^{e}R)$ for any $e \in \mathbb{N}$ (cf. Definition 1.13). Thus

$$R \text{ is strongly F-regular } \implies \mathrm{s}(R) > 0 \implies R \text{ is strongly F-regular.}$$

(2) In general, assume that R is excellent. Then R is strongly F-regular if and only if $\widehat{R}$ is strongly F-regular. (For example, this is immediate via the existence of a completely stable test element for $0 \subset E$; see Remark 3.5(1).) Also observe that $\mathrm{s}(R) = \mathrm{s}(\widehat{R})$, see [Yao, Remark 2.3(3)]. In light of (1) above, this sketches an alternate proof of the equivalence

$$R \text{ is strongly F-regular } \iff \mathrm{s}(R) > 0$$

without assuming that R is F-finite. See Theorem 1.14(2).

4. Basic results on F-rational signature

The next theorem explains why $\mathrm{r}(R)$ is called the F-rational signature of R. Its proof depends on Theorem 3.1 and Theorem 2.5.

Theorem 4.1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p. Consider the conditions:*

(1) $\mathrm{r}(R) > 0$*;*

(2) $\mathrm{r}(M) > 0$ *for all finitely generated R-modules M with* $\dim(M) = \dim(R)$*;*

(3) $\mathrm{r}(M) > 0$ *for some finitely generated R-module M;*
(4) $\widehat{R}$ *is F-rational;*
(5) R *is F-rational.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5). *If, moreover, R is excellent or there exists a common parameter (weak) test element for R and $\widehat{R}$, then* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

Proof. The implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are always true. The implication (4) $\Leftarrow$ (5) holds if there exists a common parameter (weak) test element for R and $\widehat{R}$ (the proof of [Sm2, Lemma 1.4] still works with the existence of a common parameter weak test element for R and $\widehat{R}$). This is the case when R is excellent (as completely stable weak test elements exist; cf. Theorem 1.6). It remains to show (3) $\Rightarrow$ (4) $\Rightarrow$ (2). To do this, we assume $R = \widehat{R}$ without loss of generality (cf. Observation 2.2(1)).

If $R = \widehat{R}$ is not F-rational, then there exist a system of parameters $\underline{x}$ and an element $u \in (\underline{x})^* \setminus (\underline{x})$, which will force $\mathrm{r}(M) = 0$ for all M. This proves (3) $\Rightarrow$ (4). (Also see Observation 2.2(4).)

For (4) $\Rightarrow$ (2), we assume that $R = \widehat{R}$ is F-rational and, hence, a complete domain. For any finitely generated R-module M with $\dim(M) = \dim(R)$, its torsion-free rank, say n, is positive and we have $\mathrm{r}(M) = n\,\mathrm{r}(R)$ by Observation 2.2(3). So it is enough to prove (1). Pick a system of parameters, say $\underline{x}$. By Theorem 2.5, $\mathrm{r}_R(R) = \inf\{\mathrm{e}_{\mathrm{HK}}((\underline{x}), R) - \mathrm{e}_{\mathrm{HK}}((\underline{x}, v), R) \,|\, v \notin (\underline{x})\}$. Moreover, by Theorem 3.1, there exists $0 < \delta \in \mathbb{R}$ such that, for any $J \supseteq (\underline{x})$, $\mathrm{e}_{\mathrm{HK}}((\underline{x}), M) - \mathrm{e}_{\mathrm{HK}}(J, M)$ is either 0 or $\geqslant \delta$. Finally, [HH1] (cf. Theorem 1.11) yields $\mathrm{e}_{\mathrm{HK}}((\underline{x}), R) - \mathrm{e}_{\mathrm{HK}}((\underline{x}, v), R) > 0$ for every $v \notin (\underline{x})^*$; thus, $\mathrm{e}_{\mathrm{HK}}((\underline{x}), R) - \mathrm{e}_{\mathrm{HK}}((\underline{x}, v), R) \geqslant \delta$ for every $v \notin (\underline{x}) = (\underline{x})^*$. Therefore, $\mathrm{r}_R(R) \geqslant \delta > 0$. □

We next prove some relations among the invariants $\mathrm{r}(M)$, $\mathrm{s}(M)$, $\mathrm{e}(M) = \mathrm{e}(\mathfrak{m}, M)$ and $\mathrm{e}_{\mathrm{HK}}(M) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{m}, R)$.

Lemma 4.2. *If $(R, \mathfrak{m}, k)$ is a Noetherian local ring $(R, \mathfrak{m}, k)$ of characteristic p and M is a finitely generated R-module, then*

(1) $\mathrm{s}(M) \leqslant \mathrm{r}(M) \leqslant \mathrm{e}_{\mathrm{HK}}(M)$.
(2) $\mathrm{s}(M) \leqslant \mathrm{r}(M) \leqslant \min\{\mathrm{e}_{\mathrm{HK}}(M), \mathrm{e}(M) - \mathrm{e}_{\mathrm{HK}}(M)\}$ *if $|k| = \infty$ and R is not regular.*

Proof. (1). The first inequality follows from Definition 2.1 and [Yao, Lemma 2.5(2)], while the second follows from (the proof of) [HuY, Lemma 2.1][3].

(2). The assumption $|k| = \infty$ ensures the existence of a system of parameters $\underline{x}$ such that the ideal $(\underline{x})$ is a reduction of $\mathfrak{m}$ and hence $\mathrm{e}(M) = \mathrm{e}((\underline{x}), M) = \mathrm{e}_{\mathrm{HK}}((\underline{x}), M)$[4]. The assumption that R is not regular guarantees $(\underline{x}) \subsetneq \mathfrak{m}$. Thus $\mathrm{e}_{\mathrm{HK}}((\underline{x}), M) - \mathrm{e}_{\mathrm{HK}}((\underline{x}, u), M) = \mathrm{e}((\underline{x}), M) - \mathrm{e}_{\mathrm{HK}}((\underline{x}, u), M) \leqslant \mathrm{e}((\underline{x}), M) - \mathrm{e}_{\mathrm{HK}}(\mathfrak{m}, M)$ for any $u \in ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})$. This implies $\mathrm{r}(M) \leqslant \mathrm{e}(M) - \mathrm{e}_{\mathrm{HK}}(M)$. □

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[3]The proof method of [HuY, Lemma 2.1] also works for module M. Alternatively, we may use the associativity formula to reduce it to the situation of rings.

[4]Lech's lemma says $\mathrm{e}((\underline{x}), M) = \lim_{n\to\infty} \frac{\lambda(M/(x_1^n, \ldots, x_{\dim(R)}^n)M)}{n^{\dim(R)}}$, which agrees with $\mathrm{e}_{\mathrm{HK}}((\underline{x}), M)$.

In the case where $(R, \mathfrak{m})$ is a regular local ring, one easily sees that $\mathrm{r}(R) = 1$, since $\mathrm{e}_{\mathrm{HK}}(I, R) = \lambda(R/I)$ for any $\mathfrak{m}$-primary ideal [HuY, Proposition 1.5]. Thus, for any R-module M, $\mathrm{r}(M)$ equals the torsion-free rank of M. We have a very easy lemma about $\mathrm{r}(R)$ when R is Gorenstein.

Lemma 4.3. *If $(R, \mathfrak{m}, k)$ is a Gorenstein Noetherian local ring of characteristic p, then $\mathrm{r}(M) = \mathrm{s}(M)$ for any finitely generated maximal Cohen-Macaulay R-module M. In particular, $\mathrm{r}(R) = \mathrm{s}(R)$.*

Proof. This follows from Definition 1.13 and Theorem 2.4(1). □

5. Deformation, flat extension, and localization

Given a local ring homomorphism $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ of Noetherian local rings of prime characteristic p, a finitely generated module M over R and $P \in \mathrm{Spec}(R)$, we get an S-module $M \otimes_R S$ by scalar extension and an R_P-module M_P by localization. To avoid cumbersome subscripts, we sometimes simply write $\mathrm{r}(M \otimes_R S)$, $\mathrm{r}(S/\mathfrak{m}S)$ and $\mathrm{r}(M_P)$ etc. instead of $\mathrm{r}_S(M \otimes_R S)$, $\mathrm{r}_{S/\mathfrak{m}S}(S/\mathfrak{m}S)$ and $\mathrm{r}_{S_P}(M_P)$ etc. respectively. As long as no confusion arises, we identify $u \in R$ with its image in S, even though the local ring homomorphism $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is not necessarily injective.

Let us first consider deformation. We know that if $(R, \mathfrak{m})$ is a homomorphic image of a Cohen-Macaulay ring and if, for some non-zero-divisor x, R/xR is F-rational, then so is R (cf. Theorem 1.4). This result can be described in terms of F-rational signature.

Theorem 5.1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p and $\underline{x} = x_1, \dots, x_h$ an R-regular sequence. Denote $\bar{R} = R/(\underline{x})R$. Then*

(1) $\mathrm{r}_R(R) \geqslant \mathrm{r}_{\bar{R}}(\bar{R})$;

(2) $\mathrm{r}_R(M) \geqslant \mathrm{r}_{\bar{R}}(M/(\underline{x})M)$ for any finitely generated maximal Cohen-Macaulay R-module M.

Proof. If $\mathrm{r}_{\bar{R}}(\bar{R}) = 0$, then we also have $\mathrm{r}_{\bar{R}}(M/(\underline{x})M) = 0$ (cf. Theorem 4.1) and both (1) and (2) are trivial. Therefore, we assume $\mathrm{r}_{\bar{R}}(\bar{R}) > 0$, which forces $\bar{R} = R/(\underline{x})R$ to be Cohen-Macaulay. Consequently, we assume that R is Cohen-Macaulay and, as a result, it suffices to prove (2).

By induction on h, it suffices to prove $\mathrm{r}_R(M) \geqslant \mathrm{r}_{\bar{R}}(M/xM)$ whenever $x \in \mathfrak{m}$ is a non-zero-divisor on R (and, hence, on M).

Extend x to a (full) system of parameters $x, y_2, \dots, y_{\dim(R)}$ of R and denote $\underline{y} = y_2, \dots, y_{\dim(R)}$. Also, for any q, denote $\underline{y}^{[q]} = y_2^q, \dots, y_{\dim(R)}^q$.

By the result of Theorem 2.5, it suffices to prove

$$\begin{aligned} &\inf\{\mathrm{e}_{\mathrm{HK}}((x, \underline{y}), M) - \mathrm{e}_{\mathrm{HK}}((x, \underline{y}, u), M) \mid u \notin (x, \underline{y})\} \\ &\qquad \geqslant \inf\{\mathrm{e}_{\mathrm{HK}}((\underline{y}), M/xM) - \mathrm{e}_{\mathrm{HK}}((\underline{y}, u), M/xM) \mid u \notin (x, \underline{y})\}. \end{aligned}$$

Hence, it is enough to show

$$\lim_{e \to \infty} \frac{\lambda\left(\frac{(x^q, \underline{y}^{[q]}, u^q)M}{(x^q, \underline{y}^{[q]})M}\right)}{q^{\dim(R)}} \geqslant \lim_{e \to \infty} \frac{\lambda\left(\frac{(x, \underline{y}^{[q]}, u^q)M}{(x, \underline{y}^{[q]})M}\right)}{q^{\dim(R)-1}} \qquad \text{for any } u \notin (x, \underline{y}).$$

Therefore, it suffices to prove, for any given $u \notin (x, \underline{y})$ and q,

$$\lambda\Big(\frac{(x^q, \underline{y}^{[q]}, u^q)M}{(x^q, \underline{y}^{[q]})M}\Big) \geqslant q\lambda\Big(\frac{(x, \underline{y}^{[q]}, u^q)M}{(x, \underline{y}^{[q]})M}\Big).$$

Thus, we may consider both u and q first as arbitrarily chosen and then as fixed elements. Moreover, to simplify notation, we denote $S = R/(\underline{y}^{[q]})R$ and $N = M/(\underline{y}^{[q]})M$, which are a one-dimensional Cohen-Macaulay local ring and an S-module, respectively. The classes of x and u are still denoted by x and u. Denote u^q by v. Now, to finish the proof, it remains only to establish the inequality

$$(*) \qquad \lambda_S\Big(\frac{(x^q, v)N}{(x^q)N}\Big) \geqslant q\lambda_S\Big(\frac{(x, v)N}{(x)N}\Big).$$

Since N is Cohen-Macaulay and x is a non-zero-divisor, there is an injective S-linear map $\phi : N/(x)N \to N/(x^q)N$ sending the class of z to the class of $x^{q-1}z$ for every $z \in N$. It is easy to see that (since ϕ is injective)

$$\lambda\Big(\frac{(x, v)N}{(x)N}\Big) = \lambda\Big(\phi\Big(\frac{(x, v)N}{(x)N}\Big)\Big) = \lambda\Big(\frac{(x^q, x^{q-1}v)N}{(x^q)N}\Big).$$

The inclusion $(x^q)N \subseteq (x^q, v)N$ may be filtered in the following way:

$$(x^q)N \subseteq (x^q, x^{q-1}v)N \subseteq \cdots \subseteq (x^q, x^i v)N \subseteq (x^q, x^{i-1}v)N \subseteq \cdots \subseteq (x^q, v)N.$$

There is a surjective S-map $f_i : \frac{(x^q,x^{i-1}v)N}{(x^q,x^i v)N} \to \frac{(x^q,x^{q-1}v)N}{(x^q)N}$ induced by multiplication by x^{q-i}, implying $\lambda\Big(\frac{(x^q,x^{i-1}v)N}{(x^q,x^i v)N}\Big) \geqslant \lambda\Big(\frac{(x^q,x^{q-1}v)N}{(x^q)N}\Big)$ for every $i = 1, 2, \ldots, q-1$. As a result, we have

$$\lambda\Big(\frac{(x^q, v)N}{(x^q)N}\Big) = \sum_{i=1}^{q} \lambda\Big(\frac{(x^q, x^{i-1}v)N}{(x^q, x^i v)N}\Big) \geqslant q\lambda\Big(\frac{(x^q, x^{q-1}v)N}{(x^q)N}\Big),$$

which is inequality $(*)$, and our proof is complete. □

We next want to study the behavior of F-rational signature under a local homomorphism $R \to S$. Recall that an ideal I of R is contracted from S if $I = \{r \in R \mid rS \subseteq IS\}$. We start with the special case of a local extension.

Lemma 5.2. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local ring homomorphism of Noetherian local rings of prime characteristic p. Denote $\bar{S} = S/\mathfrak{m}S$. Assume that $\dim(R) + \dim(\bar{S}) = \dim(S)$ and that $\underline{y} = y_1, y_2, \ldots, y_{\dim(\bar{S})}$ in S is such that the images of $\underline{y}$ form a (full) system of parameters of $\bar{S}$ and $(\underline{x})R$ is contracted from $S/(\underline{y})S$ for all systems of parameters $\underline{x} = x_1, \ldots, x_{\dim(R)}$ of R. Then $\mathrm{r}(M \otimes_R S) \leqslant \mathrm{r}(M)\,\mathrm{e}((\underline{y}), \bar{S})$ for any finitely generated R-module M.*

If R is an equidimensional homomorphic image of a Cohen-Macaulay ring, then the same conclusion still holds under the assumption that $(\underline{x})R$ is contracted from $S/(\underline{y})S$ for one system of parameters $\underline{x} = x_1, \ldots, x_{\dim(R)}$ of R.

Proof. The hypothesis implies that the images of $\underline{x}$ and $\underline{y}$ form a system of parameters of S. Denote $M_S = M \otimes_R S$ (as an S-module). For any $u \in ((\underline{x})R :_R \mathfrak{m}) \setminus (\underline{x})R$, we know that $u \notin (\underline{x}, \underline{y})S$ (since $(\underline{x})R$ is contracted from $S/(\underline{y})S$) and, moreover, there is an inequality as follows:

$$\lambda_S\Big(\frac{(\underline{x}, \underline{y}, u)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y})^{[q]}(M \otimes_R S)}\Big) \leqslant \lambda_S\Big(\frac{(\underline{x}, u)^{[q]} M}{(\underline{x})^{[q]} M} \otimes_R \frac{S}{(\underline{y})^{[q]} S}\Big)$$
$$\leqslant \lambda_R\Big(\frac{(\underline{x}, u)^{[q]} M}{(\underline{x})^{[q]} M}\Big) \lambda_{\bar{S}}\Big(\frac{\bar{S}}{(\underline{y})^{[q]} \bar{S}}\Big).$$

By the definition of Hilbert-Kunz multiplicity, we have

$$\lim_{e\to\infty} \frac{\lambda_S\Big(\frac{(\underline{x},\underline{y},u)^{[q]} M_S}{(\underline{x},\underline{y})^{[q]} M_S}\Big)}{q^{\dim(S)}} = \mathrm{e}_{\mathrm{HK}}((\underline{x}, \underline{y}), M_S) - \mathrm{e}_{\mathrm{HK}}((\underline{x}, \underline{y}, u), M_S),$$
$$\lim_{e\to\infty} \frac{\lambda_S\Big(\frac{(\underline{x},u)^{[q]} M}{(\underline{x})^{[q]} M}\Big)}{q^{\dim(R)}} = \mathrm{e}_{\mathrm{HK}}((\underline{x}), M) - \mathrm{e}_{\mathrm{HK}}((\underline{x}, u), M),$$
$$\lim_{e\to\infty} \frac{\lambda_{\bar{S}}\Big(\frac{\bar{S}}{(\underline{y})^{[q]}\bar{S}}\Big)}{q^{\dim(\bar{S})}} = \mathrm{e}_{\mathrm{HK}}((\underline{y}), \bar{S}) = \mathrm{e}((\underline{y}), \bar{S}).$$

Thus $\mathrm{r}(M \otimes_R S) \leqslant \mathrm{r}(M)\, \mathrm{e}_{\mathrm{HK}}((\underline{y}), \bar{S}) = \mathrm{r}(M)\, \mathrm{e}((\underline{y}), \bar{S})$, once we run through all system of parameters $\underline{x}$ of R and all $u \in ((\underline{x})R :_R \mathfrak{m}) \setminus (\underline{x})R$.

In the case where R is an equidimensional homomorphic image of a Cohen-Macaulay ring, the claim follows easily from the argument above and Theorem 2.5. ☐

The above lemma applies when $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is local and flat. In fact, the flatness gives a sharper upper bound for $\mathrm{r}(S)$. As a preparatory step, we first study certain special filtrations of an Artinian local ring.

Notation 5.3. Let $(S, \mathfrak{n}, l)$ be a Noetherian local ring and N a finitely generated S-module.

1. The type of N, denoted by $\mathrm{t}(N)$, is defined as $\mathrm{rank}_l(\mathrm{Ext}^h_S(l, N))$, where $h = \mathrm{depth}(N)$. It can be shown that $\mathrm{t}(H) = \mathrm{rank}_l((0 :_{N/(\underline{y})N} \mathfrak{n}))$ for any maximal N-regular sequence $\underline{y} = y_1, \ldots, y_h \in \mathfrak{n}$.
2. Suppose that S is Artinian. For any ideal I of R, denote $I^\sharp = (0 :_S I) = \mathrm{Ann}_S(I)$, $I^{\sharp\sharp} = (I^\sharp)^\sharp$ and $I^{\sharp\sharp\sharp} = (I^\sharp)^{\sharp\sharp}$. It is straightforward to check that $I^{\sharp\sharp\sharp} = I^\sharp$. Also, we denote by $n(S)$ the maximum $n \in \mathbb{N}$ such that there is a filtration
$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{i-1} \subsetneq I_i \subsetneq \cdots \subsetneq I_{n-1} \subsetneq I_n = S$$
with $I_i = I_i^{\sharp\sharp}$ (or, equivalently, $I_i = J_i^\sharp$ for some ideal J_i) for all $0 \leqslant i \leqslant n$.

For example, if $(S, \mathfrak{n}, l)$ is Artinian, one may choose $0 \neq w_1 \in (0 :_S \mathfrak{n})$ and let $J_1 = w_1 S$ and $I_1 = J_1^{\sharp\sharp} = (0 :_S \mathfrak{n})$. As a recursive step, suppose we have chosen I_{i-1}. Then choose any $w_i \in (I_{i-1} :_S \mathfrak{n}) \setminus I_{i-1}$ and let $J_i = I_{i-1} + w_i S$ and $I_i = J_i^{\sharp\sharp}$.

Lemma 5.4. *Let $(S, \mathfrak{n}, l)$ be an Artinian local ring which is equicharacteristic and let*

$$0 = I_0 \subsetneq J_1 \subseteq I_1 \subsetneq \cdots \subseteq I_{i-1} \subsetneq J_i \subseteq I_i \subsetneq J_{i+1} \subseteq \cdots \subseteq I_{n-1} \subsetneq J_n \subseteq I_n = S$$

be any filtration (of ideals) of S such that $J_i = I_{i-1} + w_iS$ with $(I_{i-1} :_S w_i) = \mathfrak{n}$ and $I_i = J_i^{\sharp\sharp}$ for all $1 \leqslant i \leqslant n$. Then $n \geqslant \lceil \lambda(\mathfrak{n})/\operatorname{t}(S)\rceil + 1 \geqslant \lceil \lambda(S)/\operatorname{t}(S)\rceil$. Consequently, $n(S) \geqslant \lceil \lambda(\mathfrak{n})/\operatorname{t}(S)\rceil + 1 \geqslant \lceil \lambda(S)/\operatorname{t}(S)\rceil$. In particular, $n(S) = \lambda(S)$ if S is Gorenstein.

Proof. It suffices to prove that $n \geqslant \lceil \lambda(\mathfrak{n})/\operatorname{t}(S)\rceil + 1$. As S is complete, we may assume that $l \subseteq S$, by the existence of a coefficient field. Therefore, every ideal of S is an l-vector subspace of S. For every $0 \leqslant i \leqslant n$, let $V_i = I_i^{\sharp}$. Then we have

$$S = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_{i-1} \supsetneq V_i \supsetneq \cdots \supsetneq V_{n-1} \supsetneq V_n = 0.$$

(Indeed, if $V_{i-1} = V_i$ for some $0 \leqslant i \leqslant n$, then $I_{i-1} = V_{i-1}^{\sharp} = V_i^{\sharp} = I_i$, a contradiction.) By construction, $I_1 = (0 :_S \mathfrak{n}), I_{n-1} = \mathfrak{n}$ and, hence, $V_1 = \mathfrak{n}, V_{n-1} = (0 :_S \mathfrak{n})$. For every $1 \leqslant i \leqslant n$, there exists an l-vector subspace $V_i' \subseteq V_{i-1}$ such that $V_i \oplus V_i' = V_{i-1}$. Now, as $\operatorname{rank}_l(V_1') = 1$, we only need to prove $\operatorname{rank}_l(V_i') \leqslant \operatorname{t}(S)$ for all $2 \leqslant i \leqslant n$. For any $i = 1, 2, \ldots, n$, we have $w_iV_{i-1} \subseteq (0 :_S \mathfrak{n})$, which gives rise to an l-linear map $h_i : V_i' \to (0 :_S \mathfrak{n})$ defined by $h_i(x) = w_ix$. For any $i = 1, 2, \ldots, n$ and any $x \in V_i'$, if $h_i(x) = w_ix = 0$, then $x \in \operatorname{Ann}_S(J_i) = J_i^{\sharp} = J_i^{\sharp\sharp\sharp} = (J_i^{\sharp\sharp})^{\sharp} = I_i^{\sharp} = V_i$, which implies that $x \in V_i \cap V_i' = 0$. Therefore, for every $1 \leqslant i \leqslant n$, h_i is an injective l-linear map. Hence, $\operatorname{rank}_l(V_i') \leqslant \operatorname{rank}_l(0 :_S \mathfrak{n}) = \operatorname{t}(S)$. □

Remark 5.5. Let $S = k[T, X, Y]/(T^n, X^2, XY, Y^2)$, $J = (x)$ and $I = (x, yz^{n-1})$. Then $\operatorname{t}(S) = 2$ but $\lambda(J^{\sharp\sharp}/J) = n$ and $\lambda(I^{\sharp\sharp}/I) = n - 1$ as $J^{\sharp\sharp} = I^{\sharp\sharp} = (x, y)^{\sharp} = (x, y)R$.

Theorem 5.6. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local flat ring homomorphism of Noetherian local rings of prime characteristic p. Denote $\bar{S} = S/\mathfrak{m}S$. Then, for any finitely generated R-module M, we have*

$$(\#) \qquad \operatorname{r}(M \otimes_R S) \leqslant \operatorname{r}(M) \min\Big\{ \frac{\operatorname{e}((\underline{y})\bar{S}, \bar{S})}{n(\bar{S}/(\underline{y})\bar{S})} \,\Big|\, \underline{y} \text{ is a system of parameters of } \bar{S} \Big\}.$$

In particular, we have that $\operatorname{r}(M \otimes_R S) \leqslant \operatorname{r}(M)\operatorname{t}(\bar{S})$. When $\bar{S}$ is Gorenstein, we have that $\operatorname{r}(M \otimes_R S) \leqslant \operatorname{r}(M)$.

Proof. If S is not Cohen-Macaulay, then $\operatorname{r}(M \otimes_R S) = 0$ (cf. Observation 2.2) and the claims all become trivial. Therefore, we assume that S is Cohen-Macaulay without loss of generality and, hence, R and $\bar{S}$ are Cohen-Macaulay.

Clearly, $\dim(S) = \dim(R) + \dim(\bar{S})$. Fix any system of parameters $\underline{x}$ of R. For any elements $\underline{y} = y_1, y_2, \ldots, y_{\dim(\bar{S})} \in S$ such that their images form a system of parameters for $\bar{S}$, we know that $S/(\underline{y})^{[q]}S$ is faithfully flat over R for every $q = p^e$ (cf. Theorem 1.16), which implies that, for any $u \in R\backslash(\underline{x})R$, we have that $u \notin (\underline{x}, \underline{y})S$, i.e., $(\underline{x})R$ is contracted from $S/(\underline{y})S$. Also, as $\bar{S}$ is Cohen-Macaulay, we have that $\operatorname{e}((\underline{y})\bar{S}, \bar{S})/n(\bar{S}/(\underline{y})\bar{S}) = \lambda(\bar{S}/(\underline{y})\bar{S})/n(\bar{S}/(\underline{y})\bar{S}) \leqslant \operatorname{t}(\bar{S}/(\underline{y})\bar{S}) = \operatorname{t}(\bar{S})$ by Lemma 5.4.

Consequently, we only need to prove $(\#)$. Fix any sequence $\underline{y}$ whose image is a system of parameters for $\bar{S}$. Denote $\widetilde{S} = S/(\mathfrak{m}, \underline{y})S = \bar{S}/(\underline{y})\bar{S}$ and $\widetilde{\mathfrak{n}} = \mathfrak{n}/(\mathfrak{m}, \underline{y})S$. Then $(\widetilde{S}, \widetilde{\mathfrak{n}})$ is a 0-dimensional (i.e., Artinian) local ring. Say $n(\widetilde{S}) = n$. Then there exists a filtration

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{i-1} \subsetneq I_i \subsetneq \cdots \subsetneq I_{n-1} \subsetneq I_n = \widetilde{S}$$

such that $I_i = I_i^{\sharp\sharp}$ for all $0 \leqslant i \leqslant n$. For every $i = 1, \ldots, n$, choose $w_i \in I_i$ such that $(I_{i-1} :_S w_i) = \widetilde{\mathfrak{n}}$ and, hence, there exists $s_i \in I_{i-1}^{\sharp}$ such that $0 \neq s_i w_i = v_i \in (0 : \widetilde{\mathfrak{n}})$.

Now fix a lifting of s_i, w_i, v_i, I_i up to S for every $1 \leqslant i \leqslant n$. For convenience, we continue to denote their liftings by s_i, w_i, v_i, I_i, that is, from now on, $s_i, w_i, v_i \in S$ and I_i are ideals of S for $1 \leqslant i \leqslant n$. Set $I_0 = 0$.

For any $u \in (\underline{x})R :_R \mathfrak{m}) \setminus (\underline{x})R$, we see that (as $S/(\underline{y})S$ is flat over R)

$$\frac{(\underline{x}, \underline{y}, u)S}{(\underline{x}, \underline{y})S} \cong \frac{(\underline{x}, u)R}{(\underline{x})R} \otimes_R \frac{S}{(\underline{y})S} \cong \widetilde{S},$$

which implies that $uv_i \notin (\underline{x}, \underline{y})S$ (actually $uv_i \in ((\underline{x}, \underline{y})S :_S \mathfrak{n}) \setminus (\underline{x}, \underline{y})S)$ for every $1 \leqslant i \leqslant n$.

For any $q = p^e$, we now have (recall that $n = n(\widetilde{S})$)

$$\begin{aligned}\sum_{i=1}^{n} \lambda_S\Big(\frac{(\underline{x}, \underline{y}, uI_{i-1}, uw_i)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y}, uI_{i-1})^{[q]}(M \otimes_R S)}\Big) &\leqslant \sum_{i=1}^{n} \lambda_S\Big(\frac{(\underline{x}, \underline{y}, uI_i)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y}, uI_{i-1})^{[q]}(M \otimes_R S)}\Big) \\ &= \lambda_S\Big(\frac{(\underline{x}, \underline{y}, u)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y})^{[q]}(M \otimes_R S)}\Big)\end{aligned}$$

Also notice that, for every q and every $1 \leqslant i \leqslant n$, there is an S-linear map

$$\phi_{q,i} : \frac{(\underline{x}, \underline{y}, uI_{i-1}, uw_i)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y}, uI_{i-1})^{[q]}(M \otimes_R S)} \longrightarrow \frac{(\underline{x}, \underline{y}, uv_i)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y})^{[q]}(M \otimes_R S)}$$

defined by sending a class $[z]$ to $[s_i^q z]$, which is well-defined by our choice of s_i, w_i, v_i. In particular, $\phi_{q,i}([(uw_i)^q]) = [(s_i u w_i)^q] = [(uv_i)^q]$, which means that $\phi_{q,i}$ is surjective for every q and every $1 \leqslant i \leqslant n = n(\widetilde{S}) = n(\bar{S}/(\underline{y})\bar{S})$.

Thus, for every $q = p^e$, there is an inequality

$$\begin{aligned}\sum_{i=1}^{n(\widetilde{S})} \lambda_S\Big(\frac{(\underline{x}, \underline{y}, uv_i)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y})^{[q]}(M \otimes_R S)}\Big) &\leqslant \lambda_S\Big(\frac{(\underline{x}, \underline{y}, u)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y})^{[q]}(M \otimes_R S)}\Big) \\ = \lambda_S\Big(\frac{(\underline{x}, u)^{[q]}M}{(\underline{x})^{[q]}M} \otimes_R \frac{S}{(\underline{y})^{[q]}S}\Big) &= \lambda_R\Big(\frac{(\underline{x}, u)^{[q]}M}{(\underline{x})^{[q]}M}\Big)\lambda_{\bar{S}}\Big(\frac{\bar{S}}{(\underline{y})^{[q]}\bar{S}}\Big),\end{aligned}$$

which, after we divide by $q^{\dim(S)} = q^{\dim(R)} q^{\dim(\bar{S})}$ with $q \to \infty$, implies

$$\begin{aligned}\sum_{i=1}^{n(\widetilde{S})} \big(\mathrm{e_{HK}}((\underline{x}, \underline{y}), M \otimes_R S) - \mathrm{e_{HK}}((\underline{x}, \underline{y}, uv_i), M \otimes_R S)\big) \\ \leqslant (\mathrm{e_{HK}}((\underline{x}), M) - \mathrm{e_{HK}}((\underline{x}, u), M))\, \mathrm{e}((\underline{y})\bar{S}, \bar{S}).\end{aligned}$$

This gives $n(\widetilde{S})\, \mathrm{r}(M \otimes_R S) \leqslant \mathrm{r}(M)\, \mathrm{e}((\underline{y})\bar{S}, \bar{S})$, as u is arbitrary in $((\underline{x})R :_R \mathfrak{m}) \setminus (\underline{x})R$. That is, $\mathrm{r}(M \otimes_R S) \leqslant \mathrm{r}(M)\, \mathrm{e}((\underline{y})\bar{S}, \bar{S})/n(\bar{S}/(\underline{y})\bar{S})$.

The inequality (#) now follows as we run through all $\underline{y}$ whose image is a system of parameters for $\bar{S}$. $\square$

Under the assumptions of Theorem 5.6, if $\bar{S}$ is furthermore Gorenstein and the induced map $R/\mathfrak{m} \to S/\mathfrak{n}$ is an isomorphism, then we can bound $\mathrm{r}(S)$ below.

Theorem 5.7. *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local flat ring homomorphism of Noetherian local rings of prime characteristic p. Denote $\bar{S} = S/\mathfrak{m}S$. Let M be an arbitrary finitely generated R-module.*

(1) If R is Gorenstein, then $\mathrm{r}(M \otimes_R S) \geqslant \mathrm{r}(M)\,\mathrm{r}(\bar{S})$. In particular, $\mathrm{r}(S) \geqslant \mathrm{r}(R)\,\mathrm{r}(\bar{S})$;

(2) If $\bar{S}$ is Gorenstein and the induced map $R/\mathfrak{m} \to S/\mathfrak{n}$ is an isomorphism, then $\mathrm{r}(M \otimes_R S) \geqslant \mathrm{r}(M)\,\mathrm{r}(\bar{S})$. In particular, $\mathrm{r}(S) \geqslant \mathrm{r}(R)\,\mathrm{r}(\bar{S})$;

(3) If $\bar{S}$ is regular and the induced map $R/\mathfrak{m} \to S/\mathfrak{n}$ is an isomorphism, then $\mathrm{r}(M \otimes_R S) = \mathrm{r}(M)$. In particular, $\mathrm{r}(S) = \mathrm{r}(R)$.

Proof. It suffices to prove (1) and (2) because (2) and Theorem 5.6(1) imply (3). However, all of (1), (2) and (3) will be proved from scratch in this proof. Notice that we may assume that R and $\bar{S}$ are both Cohen-Macaulay (so is S) without loss of generality (otherwise $\mathrm{r}(M) = 0$ or $\mathrm{r}(\bar{S}) = 0$ and all the claims become trivial).

Choose a system of parameters $\underline{x}$ for R. Also choose $\underline{y} \in S$ whose image is a system of parameters for $\bar{S}$. (In case $\bar{S}$ is regular, make sure the image of $\underline{y}$ is a regular system of parameters for $\bar{S}$.)

Say $((\underline{x})R :_R \mathfrak{m})/(\underline{x})R \cong k^{\oplus r} \cong \left(\frac{R}{\mathfrak{m}}\right)^{\oplus r}$ as k-vector spaces for some $1 \leqslant r \in \mathbb{N}$. Similarly, say $((\underline{y})\bar{S} :_{\bar{S}} \mathfrak{n})/(\underline{y})\bar{S} \cong l^{\oplus s} \cong \left(\frac{S}{\mathfrak{n}}\right)^{\oplus s}$ for some $1 \leqslant s \in \mathbb{N}$.

Since $S/(\underline{y})S$ is flat over R (cf. Theorem 1.16) and, hence, $S/(\underline{x}, \underline{y})S$ is flat over $R/(\underline{x})R$, we have

$$\left(0 :_{\frac{S}{(\underline{x},\underline{y})S}} \mathfrak{n}\right) = \left(0 :_{\left(0 :_{\frac{S}{(\underline{x},\underline{y})S}} \mathfrak{m}\right)} \mathfrak{n}\right) = \left(0 :_{\left(0 :_{\frac{R}{(\underline{x})R}} \mathfrak{m}\right) \frac{S}{(\underline{x},\underline{y})S}} \mathfrak{n}\right) \cong \left(0 :_{\left(k^{\oplus r} \otimes \frac{S}{(\underline{y})S}\right)} \mathfrak{n}\right)$$
$$\cong \left(0 :_{\left(\frac{R}{\mathfrak{m}} \otimes \frac{S}{(\underline{y})S}\right)^{\oplus r}} \mathfrak{n}\right) \cong \left(0 :_{\left(\frac{R}{\mathfrak{m}} \otimes \frac{S}{(\underline{y})S}\right)} \mathfrak{n}\right)^{\oplus r} \cong \left(0 :_{\frac{\bar{S}}{(\underline{y})\bar{S}}} \mathfrak{n}\right)^{\oplus r} \cong \left(\frac{S}{\mathfrak{n}}\right)^{\oplus rs},$$

which has the following implications:

(1) Under the assumption (1), we have $r = 1$. Say $\left(0 :_{\frac{R}{(\underline{x})R}} \mathfrak{m}\right) \cong k$ is generated by the image of $u \in R$. Then there is an isomorphism $\left(0 :_{\frac{\bar{S}}{(\underline{y})\bar{S}}} \mathfrak{n}\right) \cong \left(0 :_{\frac{S}{(\underline{x},\underline{y})S}} \mathfrak{n}\right)$ as l-vector spaces sending any class $[z] \mapsto [uz]$. Consequently, for any $w \in ((\underline{x}, \underline{y}) :_S \mathfrak{n}) \setminus (\underline{x}, \underline{y})$, there exists $v \in S$ whose image is in $((\underline{y})\bar{S} :_{\bar{S}} \mathfrak{n}) \setminus (\underline{y})\bar{S}$ such that $(\underline{x}, \underline{y}, uv)S = (\underline{x}, \underline{y}, w)S$.

(2, 3) If $\bar{S}$ is Gorenstein (or regular), then $s = 1$. Say $\left(0 :_{\frac{\bar{S}}{(\underline{y})\bar{S}}} \mathfrak{n}\right)$ is generated by the image of $v \in S$. (In the case $\bar{S}$ is regular, we choose $v = 1$.) Then, given the assumption that the induced map $R/\mathfrak{m} \to S/\mathfrak{n}$ is an isomorphism, there is an isomorphism $\left(0 :_{\frac{R}{(\underline{x})R}} \mathfrak{m}\right) \cong \left(0 :_{\frac{S}{(\underline{x},\underline{y})S}} \mathfrak{n}\right)$ as k-vector spaces sending any class $[z] \mapsto [zv]$. Consequently, for any $w \in ((\underline{x}, \underline{y}) :_S \mathfrak{n}) \setminus (\underline{x}, \underline{y})$, there exists

$u \in ((\underline{x})R :_R \mathfrak{m}) \setminus (\underline{x})R$ such that $(\underline{x}, \underline{y}, uv)S = (\underline{x}, \underline{y}, w)S$ (and vice versa, which is needed in proving (3)).

Therefore, in all the cases (1), (2) or (3) above, we have the following inequality (equality if $\bar{S}$ is regular). For convenience, we write $M \otimes_R S = M_S$.

$$\begin{aligned}
&\lambda_S\left(\frac{(\underline{x},\underline{y},uv)^{[q]}M_S}{(\underline{x},\underline{y})^{[q]}M_S}\right) = \lambda_S\left(\frac{(\underline{x},\underline{y},u)^{[q]}M_S}{(\underline{x},\underline{y})^{[q]}M_S}\right) - \lambda_S\left(\frac{(\underline{x},\underline{y},u)^{[q]}M_S}{(\underline{x},\underline{y},uv)^{[q]}M_S}\right)\\
&= \lambda_S\left(\frac{(\underline{x},u)^{[q]}M}{(\underline{x})^{[q]}M} \otimes_R \frac{S}{(\underline{y})^{[q]}S}\right) - \lambda_S\left(\frac{M_S}{((\underline{x},\underline{y},uv)^{[q]}M_S :_{M_S} u^q)}\right)\\
&= \lambda_R\left(\frac{(\underline{x},u)^{[q]}M}{(\underline{x})^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{(\underline{y})^{[q]}\bar{S}}\right) - \lambda_S\left(\frac{M_S}{((\underline{x},\underline{y})^{[q]}M_S :_{M_S} u^q) + v^q M_S}\right)\\
&= \lambda_R\left(\frac{(\underline{x},u)^{[q]}M}{(\underline{x})^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{(\underline{y})^{[q]}\bar{S}}\right) - \lambda_S\left(\frac{M_S}{((\underline{x})^{[q]}M_S :_{M_S} u^q) + (\underline{y})^{[q]}M_S + v^q M_S}\right)\\
&= \lambda_R\left(\frac{(\underline{x},u)^{[q]}M}{(\underline{x})^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{(\underline{y})^{[q]}\bar{S}}\right) - \lambda_S\left(\frac{M_S}{((\underline{x})^{[q]}M_S :_{M_S} u^q)} \otimes_S \frac{S}{(\underline{y},v)^{[q]}S}\right)\\
&= \lambda_R\left(\frac{(\underline{x},u)^{[q]}M}{(\underline{x})^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{(\underline{y})^{[q]}\bar{S}}\right) - \lambda_S\left(\frac{M}{((\underline{x})^{[q]}M :_{M} u^q)} \otimes_R \frac{S}{(\underline{y},v)^{[q]}S}\right)\\
&\geqslant \lambda_R\left(\frac{(\underline{x},u)^{[q]}M}{(\underline{x})^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{(\underline{y})^{[q]}\bar{S}}\right) - \lambda_R\left(\frac{M}{((\underline{x})^{[q]}M :_{M} u^q)}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{(\underline{y},v)^{[q]}\bar{S}}\right)\\
&= \lambda_R\left(\frac{(\underline{x},u)^{[q]}M}{(\underline{x})^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{(\underline{y})^{[q]}\bar{S}}\right) - \lambda_R\left(\frac{(\underline{x},u)^{[q]}M}{(\underline{x})^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{\bar{S}}{(\underline{y},v)^{[q]}\bar{S}}\right)\\
&= \lambda_R\left(\frac{(\underline{x},u)^{[q]}M}{(\underline{x})^{[q]}M}\right)\lambda_{\bar{S}}\left(\frac{(\underline{y},v)^{[q]}\bar{S}}{(\underline{y})^{[q]}\bar{S}}\right).
\end{aligned}$$

(In the case where $\bar{S}$ is regular, equality holds throughout, as $v = 1$.) Dividing both sides by $q^{\dim(S)} = q^{\dim(R)}q^{\dim(\bar{S})}$ and letting $q \to \infty$, we get

$$\begin{aligned}
&\mathrm{e}_{\mathrm{HK}}((\underline{x},\underline{y}), M \otimes S) - \mathrm{e}_{\mathrm{HK}}((\underline{x},\underline{y},w), M \otimes S)\\
&\qquad \geqslant (\mathrm{e}_{\mathrm{HK}}((\underline{x}), M) - \mathrm{e}_{\mathrm{HK}}((\underline{x},u), M))(\mathrm{e}_{\mathrm{HK}}((\underline{y})\bar{S}, \bar{S}) - \mathrm{e}_{\mathrm{HK}}((\underline{y},v)\bar{S}, \bar{S})).
\end{aligned}$$

(Equality holds if $\bar{S}$ is regular.) As w exhausts all elements in $((\underline{x},\underline{y}) :_S \mathfrak{n}) \setminus (\underline{x},\underline{y})$ we have $\mathrm{r}(M \otimes_R S) \geqslant \mathrm{r}(M)\,\mathrm{r}(\bar{S})$. (If $\bar{S}$ is regular, u will also exhaust all elements in $((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})$, implying $\mathrm{r}(M \otimes_R S) \leqslant \mathrm{r}(M)\,\mathrm{r}(\bar{S}) = \mathrm{r}(M)$.) Now (1), (2) and (3) are proved. □

In particular, under the assumption of Theorem 5.7, if R and $\bar{S}$ are both excellent and F-rational, then S is also F-rational. This is very similar to [En, Proposition 3.1].

Proposition 5.8. *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring of characteristic p, $P \in \operatorname{Spec}(R)$ a prime ideal such that $\dim(R/P) = 1$ and M a finitely generated R-module.*

Then $\mathrm{r}(M) \leqslant \mathrm{r}(M_P)\alpha(P)$*, where* $\alpha(P) := \min\big\{\, \mathrm{e}(x, R/P) = \lambda_R\big(\frac{R}{(P,x)R}\big) \mid x \in \mathfrak{m} \setminus P \big\}$*. If* $|k| = \infty$*, then* $\mathrm{r}(M) \leqslant \mathrm{r}(M_P)\,\mathrm{e}(R/P)$*, where* $\mathrm{e}(R/P) = \mathrm{e}(\mathfrak{m}, R/P)$ *denotes the Hilbert multiplicity of* R/P *considered as a one-dimensional ring.*

Proof. In case $|k| = \infty$, there exists $y \in \mathfrak{m} \setminus P$ such that $\mathrm{e}(R/P) = \mathrm{e}(y, R/P)$. Thus, it is enough to prove the first claim. If R is not a Cohen-Macaulay domain, then $\mathrm{r}(M) = 0$ and the claim is trivially true. Consequently, we may assume that R is a Cohen-Macaulay domain. Say $\dim(R) = d$ and therefore $\dim(R_P) = d-1$. It suffices to prove $\mathrm{r}(M) \leqslant \mathrm{r}(M_P)\,\mathrm{e}(x, R/P)$ for any $x \in \mathfrak{m} \setminus P$.

Fix an arbitrary $x \in \mathfrak{m} \setminus P$. There exists $\underline{x} = x_1, \ldots, x_{d-1} \in P$ such that $\underline{x}, x$ form a full system of parameters of R. Denote $I = (\underline{x})R$, so that IR_P is an ideal generated by a full system of parameters of R_P.

By Theorem 2.5, it is enough to show

$$\mathrm{r}(M) \leqslant \big(\mathrm{e_{HK}}(IR_P, M_P) - \mathrm{e_{HK}}((I,u)R_P, M_P)\big)\,\mathrm{e}(x, R/P)$$

for all $u \in R$ such that $(I :_R u)R_P = PR_P$. By replacing u with wu for some suitable $w \notin P$, we assume $(I :_R u) = P$ without loss of generality. We may further assume $u \notin (\underline{x}, x)R$. (Indeed, if $u \in (\underline{x}, x)R$, then write $u = v + u'x^t$ with $v \in I$, $u' \in R$ such that $t \in \mathbb{N}$ is the largest possible exponent among all such equations. Then $u' \notin (\underline{x}, x)R$, $(I,u)R_P = (I,u')R_P$ and, as x is regular on $R/(\underline{x})R$, we still have $(I :_R u') = P$. Replace u with u' and rename it as u.) We have

$$\begin{aligned}
&\lambda_R\Big(\frac{R}{(I,x)^{[q]}}\Big) - \lambda_R\Big(\frac{R}{(I,u,x)^{[q]}}\Big)\\
&\quad\leqslant \mathrm{e}\left(x^q, \frac{R}{I^{[q]}}\right) - \mathrm{e}\left(x^q, \frac{R}{(I,u)^{[q]}}\right)\\
&\quad= \mathrm{e}\left(x^q, \frac{(I,u)^{[q]}}{I^{[q]}}\right) = q\cdot \mathrm{e}\left(x, \frac{(I,u)^{[q]}}{I^{[q]}}\right)\\
&\quad= q\cdot \sum_{Q\in\min(R/I)} \lambda_{R_Q}\Big(\frac{(I,u)^{[q]}R_Q}{I^{[q]}R_Q}\Big)\,\mathrm{e}(x, R/Q) \qquad \text{(by the associativity formula)}\\
&\quad= q\cdot \lambda_{R_P}\Big(\frac{(I,u)^{[q]}R_P}{I^{[q]}R_P}\Big)\,\mathrm{e}(x, R/P) \qquad \text{(as } (I :_R u) = P\text{)}.
\end{aligned}$$

(Also note that, in the above, if $u \notin P$, then $\mathrm{e}\left(x^q, \frac{R}{(I,u)^{[q]}}\right) = 0$.) Dividing the above inequality by q^d and letting $q \to \infty$, we have

$$\mathrm{e_{HK}}((I,x), R) - \mathrm{e_{HK}}((I,u,x), R) \leqslant \big(\mathrm{e_{HK}}(IR_P, R_P) - \mathrm{e_{HK}}((I,u)R_P, R_P)\big)\,\mathrm{e}(x, R/P).$$

Finally, let n be the torsion-free rank of M over R. Then by Observation 2.2(3),

$$\begin{aligned}
\mathrm{r}(M) &\leqslant \mathrm{e_{HK}}((I,x), M) - \mathrm{e_{HK}}((I,u,x), M)\\
&= n\cdot \mathrm{e_{HK}}((I,x), R) - \mathrm{e_{HK}}((I,u,x), R)\\
&\leqslant n\cdot \big(\mathrm{e_{HK}}(IR_P, R_P) - \mathrm{e_{HK}}((I,u)R_P, R_P)\big)\,\mathrm{e}(x, R/P)\\
&= \big(\mathrm{e_{HK}}(IR_P, M_P) - \mathrm{e_{HK}}((I,u)R_P, M_P)\big)\,\mathrm{e}(x, R/P).
\end{aligned}$$

This completes the proof. □

Theorem 5.9. *Let $(R, \mathfrak{m}, k)$ be a Noetherian ring of characteristic p, with $P, Q \in \operatorname{Spec}(R)$ such that $P \subsetneq Q$, and M a finitely generated R-module. Then $\mathrm{r}(M_Q) \leqslant \mathrm{r}(M_P)\alpha(P,Q)$, where $\alpha(P,Q) := \min\left\{ \mathrm{e}(\underline{x}, (R/P)_Q)\right\}$, with $\underline{x}$ running through all systems of parameters of $(R/P)_Q$. If $|k| = \infty$ or $Q \subsetneq \mathfrak{m}$, then $\mathrm{r}(M_Q) \leqslant \mathrm{r}(M_P)\,\mathrm{e}((R/P)_Q)$ where $\mathrm{e}((R/P)_Q) = \mathrm{e}((Q/P)_Q, (R/P)_Q)$ denotes the Hilbert multiplicity of $(R/P)_Q$ considered as a local ring.*

Proof. If $|k| = \infty$ or $Q \subsetneq \mathfrak{m}$, then the residue field of $(R/P)_Q$ is infinite and hence there exists a system of parameters $\underline{x}$ of $(R/P)_Q$ such that $\mathrm{e}(\underline{x}, (R/P)_Q) = \mathrm{e}((R/P)_Q)$. Thus is suffices to prove the first claim. Without loss of generality, we may assume that $Q = \mathfrak{m}$. Then it suffices to prove that $\mathrm{r}(M) \leqslant \mathrm{r}(M_P)\,\mathrm{e}(\underline{x}, R/P)$ for any system of parameters $\underline{x}$ of R/P, which we proceed to do by induction on $\dim(R/P)$.

If $\dim(R/P) = 1$, the claim is proved in Proposition 5.8. Suppose that the claim is true for $\dim(R/P) < c$. Now let $\dim(R/P) = c \geqslant 2$ and write $\underline{x} = x_1, x_2, \ldots, x_c$. Also write $\underline{x}' = x_2, \ldots, x_c$ and $\Gamma = \{Q \,|\, Q \in \min(R/(P, \underline{x}')R), \mathrm{height}(Q/P) = c-1\}$. Fix a prime $P' \in \Gamma$. Then x_1 and $\underline{x}'$ are systems of parameters of R/P' and $(R/P)_{P'}$ respectively, which implies that

$$\mathrm{r}(M) \leqslant \mathrm{r}(M_{P'})\,\mathrm{e}(x_1, R/P') \leqslant \mathrm{r}(M_P)\,\mathrm{e}(\underline{x}', (R/P)_{P'})\,\mathrm{e}(x_1, R/P')$$

by the induction hypothesis. It suffices to prove that $\mathrm{e}(x_1, R/P')\,\mathrm{e}(\underline{x}', (R/P)_{P'}) \leqslant \mathrm{e}(\underline{x}, R/P)$. By [Mat2, Exercise 14.6], we have

$$\mathrm{e}(x_1, R/P')\,\mathrm{e}(\underline{x}', (R/P)_{P'}) \leqslant \sum_{Q \in \Gamma} \mathrm{e}(x_1, R/Q)\,\mathrm{e}(\underline{x}', (R/P)_Q) = \mathrm{e}(\underline{x}, R/P),$$

which completes the proof. □

Remark 5.10. Keep the notation as in Theorem 5.9.

(1) As can be seen from the proof of Theorem 5.9, a result that is possibly sharper than what is stated explicitly in Theorem 5.9 is that

$$\mathrm{r}(M_Q) \leqslant \mathrm{r}(M_P) \prod_{i=1}^{c} \alpha(P_i, P_{i-1}) = \mathrm{r}(M_P)\alpha(Q, P_1) \prod_{i=2}^{c} \mathrm{e}((R/P_i)_{P_{i-1}})$$

for any saturated chain of prime ideals $Q = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_{c-1} \supsetneq P_c = P$. In the case $|k| = \infty$ or $Q \subsetneq \mathfrak{m}$, we have $\mathrm{r}(M_Q) \leqslant \mathrm{r}(M_P) \prod_{i=1}^{c} \mathrm{e}((R/P_i)_{P_{i-1}})$.

(2) Suppose that R is a domain (e.g., $\mathrm{r}(R) > 0$), $Q = \mathfrak{m}$ and $P = 0$. Then Theorem 5.9 states that $\mathrm{r}(R) \leqslant \mathrm{r}(R_P)\alpha(0, \mathfrak{m}) = \alpha(0, \mathfrak{m})$, which is not as sharp as Lemma 4.2 since $\alpha(0, \mathfrak{m}) \geqslant \mathrm{e}_{\mathrm{HK}}(R)$. However, the estimate obtained in part (1) above could be sharper than Lemma 4.2.

6. An alternative F-rational signature

In this section, we define an alternative F-rational signature, which we call *the phantom F-rational signature* of a local ring $(R, \mathfrak{m})$. Using a result from [HoY], we prove that the phantom F-rational signature is positive if and only if R is F-rational. We refer the reader to [AHH] for the basic facts about phantom projective dimension of an R-module M, denoted $\mathrm{ppd}_R(M)$ or simply $\mathrm{ppd}(M)$.

Definition 6.1. Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p and M a finitely generated R-module.

(1) In case $(R, \mathfrak{m})$ is such that $\mathrm{ppd}(R/(\underline{x})) < \infty$ for every system of parameters $\underline{x}$ of R (e.g., R is an equidimensional homomorphic image of a Cohen-Macaulay ring or R is an equidimensional excellent ring), we define

$$\mathrm{r}'_R(M) = \inf\{\mathrm{e_{HK}}(L, M) - \mathrm{e_{HK}}(L/K, M) \mid \mathrm{ppd}(L) < \infty, \lambda(L) < \infty, 0 \neq K \subseteq L\}.$$

(2) Otherwise, we define $\mathrm{r}'_R(M) = 0$.

We call $\mathrm{r}'_R(R)$ the phantom F-rational signature of R.

Remark 6.2. First, from the definition, it is immediate to see that $\mathrm{r}(M) \geqslant \mathrm{r}'(M) \geqslant 0$ for every finitely generated R-module M.

Second, suppose that $\mathrm{ppd}(R/(\underline{x})) < \infty$ for every system of parameters $\underline{x}$ of R. Then we observe that $\mathrm{r}'_{\widehat{R}}(M \otimes_R \widehat{R}) \leqslant \mathrm{r}'_R(M)$ for any finitely generated R-module M. The reason is that any R-module of finite length and of finite phantom projective dimension remains so when considered as an $\widehat{R}$-module.

It turns out that $\mathrm{r}(M)$ and $\mathrm{r}'(M)$ behave quite similarly. In particular, the positivity of $\mathrm{r}'(R)$ characterizes F-rationality, in the following sense.

Theorem 6.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p. Let M be a finitely generated R-module. Consider the conditions:*

(1) $\mathrm{r}(R) > 0$;
(1′) $\mathrm{r}'(R) > 0$;
(2) $\mathrm{r}(M) > 0$ *for some* M;
(2′) $\mathrm{r}'(M) > 0$ *for some* M;
(3) $\mathrm{r}(M) > 0$ *for every* M *with* $\dim(M) = \dim(R)$;
(3′) $\mathrm{r}'(M) > 0$ *for every* M *with* $\dim(M) = \dim(R)$;
(4) $\widehat{R}$ *is F-rational;*
(5) R *is F-rational.*

Then $(1) \Leftrightarrow (1') \Leftrightarrow (2) \Leftrightarrow (2') \Leftrightarrow (3) \Leftrightarrow (3') \Leftrightarrow (4) \Rightarrow (5)$. *If, moreover, R is excellent or there exists a common weak parameter test element for R and $\widehat{R}$, then* $(1) \Leftrightarrow (1') \Leftrightarrow (2) \Leftrightarrow (2') \Leftrightarrow (3) \Leftrightarrow (3') \Leftrightarrow (4) \Leftrightarrow (5)$.

Proof. The implication $(4) \Rightarrow (5)$ is always true. And $(4) \Leftarrow (5)$ holds if there exists a common parameter test element for R and $\widehat{R}$, which is the case when R is excellent. It remains to show $(1) \Leftrightarrow (1') \Leftrightarrow (2) \Leftrightarrow (2') \Leftrightarrow (3) \Leftrightarrow (3') \Leftrightarrow (4)$.

If $\widehat{R}$ is not F-rational then $\mathrm{r}_R(M) = 0$, which forces $\mathrm{r}'_R(M) = 0$ for every finitely generated R-module M (see Theorem 4.1 and Remark 6.2). Thus, any single one of (1), (1′), (2), (2′), (3), or (3′) will imply (4).

It is straightforward to see $(3') \Rightarrow (3) \Rightarrow (2)$, $(3') \Rightarrow (2')$ and $(3') \Rightarrow (1') \Rightarrow (1)$; cf. Remark 6.2. So it remains to prove $(4) \Rightarrow (3')$ in order to complete the proof. For this, we assume that $\widehat{R}$ is F-rational. Thus, R is Cohen-Macaulay and, by Remark 6.2, $\mathrm{r}'_{\widehat{R}}(M \otimes_R \widehat{R}) \leqslant \mathrm{r}'_R(M)$ for any finitely generated R-module M. As a result, we may assume that R is a complete F-rational domain (hence Cohen-Macaulay). But now it is enough to show $\mathrm{r}'(R) > 0$ since $\mathrm{r}'(M) = \mathrm{rank}_R(M)\,\mathrm{r}'(R)$, where $\mathrm{rank}_R(M)$ denotes the torsion-free rank of M over R.

Choose c, $c' \in R^\circ$ and $q'' = p^{e''}$ to be as in Theorem 3.3. In [Ab], it is shown that $0^*_L = 0$ for every R-module L such that $\mathrm{ppd}_R(L) < \infty$. From [HoY, Corollary 3.3], there is a test exponent, say q''', for cc' and all R-modules L such that $\mathrm{ppd}_R(L) < \infty$ and $\lambda(L) < \infty$. By Theorem 3.3, $\mathrm{e}_{\mathrm{HK}}(L, R) - \mathrm{e}_{\mathrm{HK}}(L/K, R) \geqslant 1/(q''q''')^{\dim(R)}$ for all R-modules L with $\mathrm{ppd}_R(L) < \infty$ and $\lambda(L) < \infty$ and for all K with $0 \neq K \subseteq L$. By Definition 6.1, we see $\mathrm{r}'(R) \geqslant 1/(q''q''')^{\dim(R)} > 0$, completing the proof. □

7. Some examples

In this section, we compute some examples, starting with a lemma.

Lemma 7.1. *Let $S = K[x_1, \dots, x_d]$, a polynomial ring over a field K, graded over $\mathbb{N}^d$, such that $\deg(x_1^{n_1} \cdots x_d^{n_d}) = (n_1, \dots, n_d)$. Let R be a K-subalgebra of S generated by finitely many monomials. Let H denote a $\mathbb{Z}^d$-graded R-module such that each graded component of H is either 0 or isomorphic to K as a K-vector space. Let $v_1, \dots, v_k$ be finitely many homogeneous elements of H each of which is annihilated by a power of the homogeneous maximal ideal $\mathfrak{m}$ of R. For any $a = (a_1, \dots, a_k) \in K^k$, denote $v(a) := a_1v_1 + \cdots + a_kv_k$. Then the minimum value of $\dim_K Rv(a)$, for $v(a) \neq 0$, can be achieved when $a = (0, \dots, 0, a_\lambda, 0, \dots, 0)$ for some $0 \leqslant \lambda \leqslant k$.*

Proof. Suppose that the minimum dimension is achieved on $Rv(a)$. Any v_i with coefficient $a_i = 0$ may be omitted from consideration, and so we may suppose that all the a_i are non-zero. We may further assume that the v_i are in distinct homogeneous components: otherwise, one is a K-scalar multiple of another and may be omitted.

Linearly order $\mathbb{Z}^d$ lexicographically. We may renumber the v_i so that $\deg(v_1) > \deg(v_2) > \cdots > \deg(v_k)$. Let $\mu_1, \dots, \mu_h$ be the monomials in R that do not kill v_1. Similarly, we may renumber the μ_i so that $\deg(\mu_1) < \deg(\mu_2) < \cdots < \deg(\mu_h)$. Then the $\mu_i v_1$ are in distinct homogeneous components of H and are non-zero. It follows that they are linearly independent, and that $\dim_K Rv_1 = h$. To complete the proof, it will suffice to show that the $\dim_K Rv(a) \geqslant h$. For this it is enough to show that the elements $\mu_i v(a)$ are linearly independent over K for $1 \leqslant i \leqslant h$. In $\mu_i v(a)$ the highest degree component corresponds to $a_1\mu_i v_1$, as the other terms have strictly lower degree. But this implies linear independence: otherwise, one of the $\mu_i v(a)$ is a linear combination of its predecessors $\mu_j v(a)$ for $j < i$, which is impossible, since $\mu_i v_1$ has strictly higher degree than any term in any $\mu_j v(a)$ for $j < i$. □

Corollary 7.2. *Let R, $\mathfrak{m}$, H, $v_1, \dots, v_k$, $a \in K^k$ and $v(a)$ be as in the lemma above, and let d be an integer (d is the Krull dimension of R in the applications). Let H have an action of Frobenius, denoted F, such that for homogeneous elements u, $\deg(F(u)) = p\deg(u)$. Note that if u is killed by $\mathfrak{m}^N$, then $F(u)$ is killed by $(\mathfrak{m}^N)^{[p]}$ and so is also killed by a power of $\mathfrak{m}$. Suppose that $L(a) := \lim\limits_{e\to\infty} \dfrac{\dim_K(RF^e(v(a)))}{q^d}$ exists for every a. Then $\inf\{L(a) \mid v(a) \neq 0\}$ can be achieved at $a = (0, \dots, 0, a_\lambda, 0, \dots, 0)$, or equivalently at $a = e_\lambda = (0, \dots, 0, 1, 0, \dots, 0)$, for some $0 \leqslant \lambda \leqslant k$.*

Proof. For every choice of e, the least value of $\dim_K(RF^e(v(a)))$ is assumed when $a = (0, \dots, 0, a_{\lambda_e}, 0, \dots, 0)$ for some $0 \leqslant \lambda_e \leqslant k$, since $F^e(v(a)) = a_1^q F^e(v_1) + \cdots + a_k^q F^e(v_k)$ and hence Lemma 7.1 applies. Some λ occurs as the value of λ_e for infinitely many e. This choice of λ clearly gives the infimum. $\square$

First, we consider Segre products. In the sequel, if θ is a tuple of non-negative real numbers, we write $|\theta|$ for the sum of its entries.

Example 7.3. Let K be a field and consider the Segre product R of a polynomial ring in r variables $x = x_1, \dots, x_r$ over K and a polynomial ring in s variables $y = y_1, \dots, y_s$ over K, where $2 \leqslant r \leqslant s$. The dimension of this ring is $r + s - 1$. Let $S = K[x, y]$. Let $\mathfrak{m}$ be the homogeneous maximal ideal of R, which becomes $PQ = P \cap Q$ when expanded to S, where $P = (x)S$ and $Q = (y)S$. From the Mayer-Vietoris sequence for local cohomology, $\mathrm{H}^{r+s-1}_{\mathfrak{m}}(S) = \mathrm{H}^{r+s-1}_{P\cap Q}(S) \cong \mathrm{H}^{r+s}_{P+Q}(S) =: H$, which we view as the K-span of the monomials in x, y with strictly negative exponents. Every monomial element of H can be written $x^{-i}y^{-j}$ where i is an r-tuple of positive integers, and j is an s-tuple of positive integers. Because of the Reynolds operator $\rho : S \to R$, which fixes the monomials $x^i y^j$ such $|i| = |j|$ and kills the monomials such that $|i| \neq |j|$, we may view $\mathrm{H}^{r+s-1}_{\mathfrak{m}}(R) \subseteq \mathrm{H}^{r+s-1}_{\mathfrak{m}}(S) \cong \mathrm{H}^{r+s}_{P+Q}(S)$ as the span of those $x^{-i}y^{-j}$ such that $|i| = |j|$.

The socle of $\mathrm{H}^{r+s-1}_{\mathfrak{m}}(R)$ has a K-basis consisting of all monomials μ_i of the form $x^{-i}y_1^{-1} \cdots y_s^{-1}$ where the entries of i are positive and $|i| = s$. Decreasing each entry of i by 1 gives a bijection with all non-negative monomials of degree $s - r$ in x.

Let $i = (i_1, \dots, i_r)$, where $|i| = s$. We want to compute

$$V_i := \lim_{q\to\infty} \frac{\lambda(R\mu_i^q)}{q^{r+s-1}}.$$

The length $\lambda(R\mu_i^q)$ is the number of monomials that are strictly negative and are R-multiples of μ_i^q. Then the length is the number of lattice points $(a, b) \in \mathbb{N}^r \times \mathbb{N}^s$ such that $1 \leqslant a_h \leqslant i_h q$ for $1 \leqslant h \leqslant r$, $1 \leqslant b_k \leqslant q$ for $1 \leqslant k \leqslant s$, and such that $|a| = |b|$. For the purpose of computing the limit we may replace the length by the $(r + s - 1)$-volume of the region in $\mathbb{R}^{r+s}$ described by the restrictions above. If we scale the region down by a factor of $1/q$, its volume is divided by q^{r+s-1}.

Therefore, the limit V_i is the volume of the region in $\mathbb{R}^{r+s}$ consisting of points (u, v) that satisfy $0 \leqslant u_h \leqslant i_h$ for all h, $0 \leqslant v_k \leqslant 1$ for all k, and that satisfy the linear equation $|u| = |v|$. By Corollary 7.2, the F-rational signature $\mathrm{r}(R_{\mathfrak{m}})$ of R localized at its maximal homogeneous ideal $\mathfrak{m}$ is the least such volume V_i for choices of positive $i = (i_1, \dots, i_r)$ with $|i| = s$.

We shall show that F-rational signature in the case where $r = 2$, $s \geqslant 2$ is

$$\mathrm{r}(R) = \left(1 - \frac{2}{(s+1)!}\right)\sqrt{2}.$$

We need to minimize the V_i. Let $v = (v_1, \dots, v_s)$. Note that, by symmetry, we only need to calculate V_i for $i = (a, b)$ with $b = s - a$ when $a \leqslant b$. Hence, we may assume that $1 \leqslant a \leqslant \left\lfloor \frac{s}{2} \right\rfloor$. Let $L_a(v)$ be the length of the line segment in the (u_1, u_2)-plane defined by the conditions $u_1 \in [0, a]$, $u_2 \in [0, b]$ (which give a rectangle) and $u_1 + u_2 = |v|$. Note that $|v| \in [0, s]$. It is easy to check that

$$L_a(v) = \begin{cases} |v|\sqrt{2} & \text{if} \quad 0 \leqslant |v| \leqslant a, \\ a\sqrt{2} & \text{if} \quad a \leqslant |v| \leqslant b, \\ (s - |v|)\sqrt{2} & \text{if} \quad b \leqslant |v| \leqslant s. \end{cases}$$

Let $C := [0, 1]^s$. The required $(s+1)$-dimensional volume is then $\int_C L_a(v)\, dV$. The integral can be evaluated by breaking up the cube into the three regions $C_{a,k}$, $k \in \{1, 2, 3\}$, obtained by imposing the additional condition that $0 \leqslant |v| \leqslant a$ if $k = 1$, that $a \leqslant |v| \leqslant b$ if $k = 2$, and that $b \leqslant |v| \leqslant s$ if $k = 3$.

For $k = 1$, we get the $\sqrt{2}\int_{C_{a,1}} |v|\, dV$. This is the sum of s integrals where the integrand is v_j, $1 \leqslant j \leqslant s$. These s integrals are equal by symmetry, and so their sum is $s\sqrt{2}\int_{C_{a,1}} v_1\, dV$. Let $A_a(v_1)$ denote the $(s-1)$-dimensional volume of the intersection of $C_{a,1}$ with the linear space obtained by requiring that the first coordinate be v_1. Then $s\sqrt{2}\int_{C_{a,1}} v_1\, dV = s\sqrt{2}\int_0^a A_a(v_1)v_1\, dv_1$. Since $A_a(v_1)$ is the volume of an $(s-1)$ simplex with $s-1$ mutually orthogonal legs of length $a - v_1$, its $(s-1)$-dimensional volume is $\frac{1}{(s-1)!}(a - v_1)^{s-1}$, so that the integral becomes $s\sqrt{2}\int_0^a \frac{1}{(s-1)!}(a - v_1)^{s-1}v_1\, dv_1$. The substitution $w = a - v_1$ yields

$$s\sqrt{2}\frac{1}{(s-1)!}\int_0^a w^{s-1}(a-w)dw = s\sqrt{2}\frac{1}{(s-1)!}\left(\frac{a^s a}{s} - \frac{a^{s+1}}{s+1}\right) = \frac{sa^{s+1}\sqrt{2}}{(s+1)!}.$$

The second integral is $\sqrt{2}$ times the volume of that part of the cube where $a \leqslant |v| \leqslant s - a$. We can get this volume by subtracting the volumes of the part of the cube where $|v| \leqslant a$ and the part of the cube where $|v| \geqslant s - a$. The latter can be described as the part where $\sum_{j=1}^{s}(1 - v_j) \leqslant a$, so these are both k-simplices with k mutually perpendicular legs of length a, and both have volume $\frac{a^s}{s!}$. Thus, the volume of $C_{2,a}$ is $1 - \frac{2a^s}{s!}$, and the value of the integral over C_2 is $\left(1 - \frac{2a^s}{s!}\right)\sqrt{2}$.

Finally, if we make the substitution $w_j = 1 - v_j$ for $1 \leqslant j \leqslant s$ we see that the integral over $C_{3,a}$ is the same as the integral over $C_{1,a}$, and the total volume is

$$\int_C L_a(v)\, dV = \left(\frac{2sa^{s+1}}{(s+1)!} + 1 - \frac{2a^s}{s!}\right)\sqrt{2} = \left(1 + \frac{2a^s(sa - (s+1))}{(s+1)!}\right)\sqrt{2}.$$

To obtain the F-rational signature, we must find the minimal value for $a \in \left[1, \left\lfloor \frac{s}{2} \right\rfloor\right]$. This occurs when $a = 1$, and so the F-rational signature is $\left(1 - \frac{2}{(s+1)!}\right)\sqrt{2}$.

Next, we consider Veronese subrings.

Example 7.4. Let S denote the polynomial ring in variables $x_1, \dots, x_d$ over a field K of characteristic $p > 0$, with $d \geqslant 1$. Let $n \geqslant 1$ and let R denote the n-Veronese subring of S spanned by all monomials whose degree is a multiple of n. (Note that R is regular when $d = 1$ or $n = 1$; therefore, the non-trivial case is when $d \geqslant 2$ and $n \geqslant 2$.) Consider the system of parameters $x_1^n, \dots, x_d^n$ for R. We first want to describe the socle in R/I, with $I = (x_1^n, \dots, x_d^n)R$. For any monomial $\mu \in S$, it is routine to see that μ represents a non-zero element of the socle of R/I if and only if the following three conditions hold:

$$\mu \mid (x_1 \cdots x_d)^{n-1} \text{ in } S, \qquad d(n-1) - \deg(\mu) < n, \qquad n \mid \deg(\mu).$$

This equivalently says that $\mu = (x_1 \cdots x_d)^{n-1}/(x_1^{a_1} \cdots x_d^{a_d})$ for some $a = (a_1, \dots, a_d) \in \mathbb{N}^d$ with $|a| \leqslant n-1$ and $d(n-1) \equiv |a| \mod n$. Let r be the least *positive* integer such that $d \equiv r \mod n$. Now $d(n-1) \equiv |a| \mod n$ if and only if $|a| = n - r$. Thus, the dimension of the socle of R/I (as a vector space over K) will be the number of monomials in S of degree $n - r$. Note that the ring R is Gorenstein precisely when $r = n$, which simply means $n \mid d$.

Given any $a = (a_1, \dots, a_d) \in \mathbb{N}^d$ with $|a| = n - r$, we consider the monomial $\mu := \mu_a = (x_1 \cdots x_d)^{n-1}/(x_1^{a_1} \cdots x_d^{a_d})$ and calculate

$$\lim_{q\to\infty} \lambda((I^{[q]} :_R \mu^q)/I^{[q]})/q^d = \lim_{q\to\infty} \lambda(R/(I^{[q]} :_R \mu^q))/q^d.$$

Note that $(I^{[q]} :_R \mu^q) = (I^{[q]}S :_S \mu^q) \cap R$, for if $s \in R$ is such that $s\mu^q \in I^{[q]}S$, then $s\mu^q \in I^{[q]}S \cap R = I^{[q]}$, since R is a direct summand of S as an R-module; the other inclusion is obvious. But the flatness of the Frobenius endomorphism of S implies that

$$(I^{[q]}S :_S \mu^q) = (I :_S \mu)^{[q]} = (x_1^{a_1+1}, \dots, x_d^{a_d+1})^{[q]}S = (x_1^{q(a_1+1)}, \dots, x_d^{q(a_d+1)})S.$$

Consequently $(I^{[q]} :_R \mu^q) = (I^{[q]}S :_S \mu^q) \cap R$ is spanned by all monomials whose degree is a multiple of n such that for some i, the exponent on x_i is at least $q(a_i + 1)$. Thus, $R/(I^{[q]} :_R \mu^q)$ is spanned by all monomials such that the exponents of x_i are at most $q(a_i + 1) - 1$ for all i and the total degree is congruent to 0 modulo n. If we ignore the congruence condition on the degree modulo n, we get

$$V_a := \lim_{q\to\infty} \lambda(S/(I^{[q]}S :_S \mu^q))/q^d = \prod_{i=1}^{d} (a_i + 1).$$

With the congruence condition that the total degree be divisible by n, we get

$$\lim_{q\to\infty} \lambda(R/(I^{[q]} :_R \mu^q))/q^d = V_a/n = \left(\prod_{i=1}^{d} (a_i + 1) \right) /n.$$

By (an equivalent version of) Corollary 7.2, the F-rational signature $\mathrm{r}(R_{\mathfrak{m}})$ of R localized at its maximal homogeneous ideal $\mathfrak{m}$ is the smallest value of V_a/n as a varies through all d-tuples of non-negative integers with $|a| = n - r$, which is

$$\mathrm{r}(R_{\mathfrak{m}}) = (n - r + 1)/n,$$

achieved at $a = (0, \ldots, 0, n-r, 0, \ldots, 0)$. For example, $\mathrm{r}(R_{\mathfrak{m}}) = 1$ when $r = 1$ (e.g., $d = 3$ and $n = 2$). When $r = n$ (i.e., when R is Gorenstein), $\mathrm{r}(R_{\mathfrak{m}}) = 1/n = \mathrm{s}(R_{\mathfrak{m}})$. When $d \leqslant n$, we have $r = d$ and, hence, $\mathrm{r}(R_{\mathfrak{m}}) = (n - d + 1)/n$.

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