Qualifying Examination in Algebra at GSU – Prepared by F. Enescu and Y. Yao

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Please write your name on every page.

(1) Show that any field extension of degree 2 is normal. Is it necessarily separable? Prove or give a counterexample.

(2) Let E be the splitting field of $X^3 - 2$ over $\mathbf{Q}$. Find the Galois group of E over $\mathbf{Q}$. Please explain your answer fully.

(3) Show that there is no simple group of order 80.

(4) Let M be a $\mathbb{Z}$-module that is isomorphic to $\mathbb{Z}^5$ modulo the $\mathbb{Z}$-submodule generated by the column vectors of the following matrix

$$A = \begin{pmatrix} -2 & -8 & 6 & 4 \\ 3 & 9 & -6 & -3 \\ 4 & 16 & 12 & 16 \\ 7 & 19 & -12 & -5 \\ 6 & -6 & 12 & 18 \end{pmatrix}$$

(a) Find the Smith normal form of A over $\mathbb{Z}$.
(b) Find the invariant factors of A over $\mathbb{Z}$.
(c) Find the elementary divisors of A over $\mathbb{Z}$.
(d) Up to isomorphism, write M as a direct sum of cyclic $\mathbb{Z}$-modules.

(5) Let $n \geq 1$ be an integer. Show that every abelian group of cardinality n is cyclic **if and only if** n is square-free.

(6) Show that the elements $2(1 + i\sqrt{5})$ and 6 do not have a greatest common divisor in $\mathbb{Z}[i\sqrt{5}]$. (A greatest common divisor for a and b in a domain R is an element d of R such that it divides both a and b and has the property that any other common divisor c of a and b divides d.)